Finding Hamiltonian Circuits in Arrangements of Jordan Curves is NP-Complete

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ABSTRACT

Let $A = \{C_1, C_2, \ldots, C_n\}$ be an arrangement of Jordan curves in the plane lying in general position, i.e., every curve properly intersects at least one other curve, no two curves touch each other and no three meet at a common intersection point. The Jordan-curve arrangement graph of A has as its vertices the intersection points of the curves in A, and two vertices are connected by an edge if their corresponding intersection points are adjacent on some curve in A. We further assume A is such that the resulting graph has no multiple edges. Under these conditions it is shown that determining whether Jordan-curve arrangement graphs are Hamiltonian is NP-complete.

KEYWORDS

NP-completeness; Hamiltonian circuit; arrangements of Jordan curves; computational complexity; computational geometry

1 Introduction

A *Hamiltonian circuit* in a graph is a circuit which passes through every vertex of the graph exactly once. The *Hamiltonian circuit problem* asks whether there exists at least one Hamiltonian circuit in a given graph. There have been at least three approaches taken in the past towards the study of Hamiltonian circuits. In one approach sufficient conditions are sought for which graphs are Hamiltonian. For example, it is known that all 4-connected triangulated graphs [15], 4-connected planar graphs [14], [3] and 1-sail line arrangement graphs [4] are Hamiltonian. Also, the visibility graphs of sets of line segments with the property that the line segments are of unit length whose endpoints have integer coordinates, are Hamiltonian [10]. A related computational question concerns how fast we can find a Hamiltonian circuit in a Hamiltonian graph. For any 4-connected planar graph G with n vertices, a Hamiltonian circuit in G can be found in $O(n^3)$ time [7]. If only the vertices where a turn is made need be reported (a *streamlined* Hamiltonian circuit) then a Hamiltonian circuit for 1-sail line arrangement graphs can be found in $\Theta(n \log n)$ time, where n is the number of lines in the arrangement [5]. A second approach is to find restricted classes of graphs for which we can determine in polynomial time whether or not instances of such graphs admit a Hamiltonian circuit. For example if each line segment of a set of n disjoint line segments in the plane has at least one of its end points on the convex hull of the set, it can be determined in $O(n \log n)$ time whether the set admits a Hamiltonian circuit through its endpoints such that it is a simple polygon and uses every line segment exactly once [12]. The third approach to the Hamiltonian circuit problem has been to search for restricted classes of graphs for which the problem is NP-complete. For example, the Hamiltonian circuit problems for general graphs [8], for 3-regular 3-connected planar graphs [6], and for 3-regular bipartite planar graphs [1] are known to be NP-complete. (A graph is said to be 3-*regular* if each vertex of the graph has degree 3.) Also, in the line segment problem discussed above, if the convex hull restriction is removed and line segments are allowed to touch at their end points it is NP-complete to determine if they admit a simple Hamiltonian circuit [11]. One of the results on a related problem is the NP-completeness of the edge Hamiltonian path problem for bipartite graphs [9].

Recently several different classes of arrangement graphs have been introduced [4]. For example, an arrangement of n lines in general position (no two parallel and no three concurrent) defines a set of intersection points joined by edges. The graph whose vertices are the intersection points and whose edges are the segments of the lines between adjacent intersection points, is called a *line-arrangement graph*. Hazel Everett [5] has shown that not all line-arrangement graphs are Hamiltonian. On the other hand the great-circle-arrangement graph on the sphere (obtained in a similar manner from a set of great circles on the sphere in general position) has been recently shown to be Hamiltonian by Bruce Reed [13]. Using stereographic projection of the sphere onto the plane (the light source is at the furthest point on the sphere from the plane and is not on any circle), the great-circle arrangement maps to an arrangement of Jordan curves satisfying the following two conditions: (i) every Jordan curve intersects every other in exactly two points and (ii) there exists a point in the plane that is contained in every curve. From Reed's result, we can show easily that *Jordan-curve arrangement graphs* satisfying these conditions are Hamiltonian. In this note we show that if the above two conditions are removed, the Hamiltonian circuit problem is NP-complete. More precisely, we establish the NP-completeness of the Hamiltonian circuit problem for the class of Jordan-curve arrangement graphs with no multi-edges. This class is properly contained in the class of 4-regular graphs. Therefore our result is strictly stronger than the NP-completeness result for 4-regular planar graphs.

2 Definitions and Results

A *Jordan curve* is a curve that partitions the plane into two disjoint regions, a bounded region referred to as the interior and an unbounded region called the exterior, which are separated by the curve. Let C_i and C_j denote two Jordan curves. Let $A = \{C_1, C_2, \ldots, C_n\}$ be an *arrangement of Jordan curves*. The set of intersection points of C_i and C_j is denoted by $I(C_i, C_j)$. In this paper, we assume that no two curves are coincident in a non-zero measure intersection. We further assume that (i) every curve intersects at least one other curve, (ii) no two curves touch each other, i.e., each pair of curves intersect properly if they intersect at all and (iii) no three curves are concurrent, i.e., share a common intersection point. Let $V(A) = \{v | v \in I(C_i, C_j)$ such that $C_i, C_j \in A\}$. The *Jordan-curve arrangement graph* of an arrangement A is the graph $G_a = (V_a, E_a)$ such that (i) $V_a = V(A)$ and (ii) edges in E_a are formed by curves of A (see Fig. 1). Note that a Jordan-curve arrangement graph may contain multi-edges. (For example, the Jordan-curve arrangement graph in Fig. 1-(b) contains two pairs of multi-edges.)

Theorem 1. *The Hamiltonian circuit problem for Jordan-curve arrangement graphs with no multi-edges is NP-complete.*

Remark. Since every Jordan-curve arrangement graph is a 4-regular planar graph, the Hamiltonian circuit problem for 4-regular planar graphs is also NP-complete. The class of Jordan-curve arrangement graphs with no multi-edges is properly contained in the class of 4-regular planar graphs, i.e., there exist 4-regular planar graphs G with no multi-edges such that G cannot be formed by any arrangement of Jordan curves (see Fig. 2). It should be noted that the class of Jordan-curve arrangement graphs with no multi-edges is contained in the class of *multi*-graphs formed by arrangements of Jordan curves lying in *arbitrary* position. As a corollary of Theorem 1, the Hamiltonian circuit problem for arrangement graphs with multi-edges formed by Jordan curves lying in arbitrary position is NP-complete.

Proof of Theorem 1. Since the Hamiltonian circuit problem for general graphs is in NP [8], the problem for Jordan-curve arrangement graphs with no multi-edges is also in NP. It is known that the Hamiltonian circuit problem for 3-regular planar graphs with no multi-edges is NP-complete [6]. We reduce each 3-regular planar graph G with no multi-edges to a Jordan-curve arrangement graph G_a with no multi-edges such that G is Hamiltonian if and only if G_a is Hamiltonian. The overview of the proof is as follows. Starting with G, (i) we construct 4-regular planar graph G_1 with *multi*-edges and (ii) we then construct 4-regular planar graph G_a with *no* multi-edges. (iii) We prove that G is Hamiltonian if and only if G_a is Hamiltonian (Lemma 1), and (iv) we then prove that G_a is a Jordan-curve arrangement graph (Lemma 2). Note that the input to the problem is not an arrangement but a graph. Using the coloring algorithm given in the proof of Lemma 2, we can determine in polynomial time whether the given graph is a Jordan-curve arrangement graph. It is known that there is a linear-time algorithm for generating a planar embedding of a planar graph [2]. Thus, we can assume that the planar graphs in this paper are embedded in the plane without any crossing edges.

Construction of G_1: Let v be a vertex of G, and let x, y, and z be the three neighbors of v (see Fig. 3-(a)). We first replace each edge, say (v, x), of G by two vertices v_x, x_v and add a pair of multi-edges between v_x and x_v (see Fig. 3-(b)). We then replace each vertex v of G by three edges, (v_x, v_y), (v_y, v_z), and (v_z, v_x). We call the subgraph induced by these three edges a *triangle*. We denote the resulting graph by $G_1 = (V_1, E_1)$. G_1 can be obtained by locally replacing each vertex and each edge of G with a triangle and a pair of multi-edges, respectively. Thus, G_1 can also be embedded in the plane without any crossing edges. Now each vertex of G_1 has two edges of a triangle and a pair of multi-edges. Therefore, G_1 is a 4-regular planar graph with multi-edges. Furthermore, G_1 can be constructed from G in polynomial time.

Construction of G_a: The basic idea is to add four vertices and four edges to each pair of multi-edges (see Figs. 4 and 5). We first find a vertex subset $S \subseteq V_1$ such that (i) exactly one vertex of each pair of multi-edges is in S and (ii) at least one vertex of each triangle is in S. (We will show how to find such an S later.) Suppose that a vertex a of G_1 is in S (see Fig. 4-(a)). Let e_1, e_2, e_3, and e_4 be the edges incident to a in clockwise order in the plane. For $1 \leq i \leq 4$, we "divide" edge e_i into two edges by adding a new vertex a_i on e_i (see Fig. 4-(b)). We then add four edges (a_1, a_2), (a_2, a_3), (a_3, a_4), and (a_4, a_1). We call a subgraph induced by these four edges a *circle*. By applying the above procedure to each vertex in S, we obtain G_a (see Fig. 5-(b)). Now G_a has no multi-edges. (For example, the 3-regular graph shown in Fig. 6-(a) is transformed into the 4-regular graph in Fig. 7-(b). The reason for using the set $S \subseteq V_1$ rather than using all of V_1 is given in the proof of Lemma 1.)

It remains to show how to construct $S \subseteq V_1$ in polynomial time. We first construct a vertex set $S_1 \subseteq S$ such that (i) *at most* one vertex of each pair of multi-edges is in S_1 and (ii) *exactly* one vertex of each triangle is in S_1 (see Fig. 7-(a)). Note that there are pairs of multi-edges in G_1 neither of whose vertices are in S_1. By adding one arbitrary vertex of each such pair to S_1, we obtain S. The construction of S_1 is as follows. We construct a *directed* subgraph $D = (V, E_D)$ in the original graph $G = (V, E)$ such that all of D's vertices have out-degree one (see Figs. 5-(a) and 6-(c)). Recall that vertices and edges in G were replaced by triangles and pairs of multi-edges in G_1, respectively (see Figs. 3-(b) and 7-(a)). A vertex v_x of G_1 is in S_1 if and only if there exists a directed edge (v, x) in E_D (see Figs. 3-(b) and 5-(a)), i.e., we can obtain vertex set S_1 from edge set E_D by replacing each $(v, x) \in E_D$ with vertex $v_x \in S_1$. $D = (V, E_D)$ can be constructed as follows. We first find an *undirected* spanning tree, say $T = (V, E_T)$, in G (see Fig. 6-(b)). We choose an arbitrary vertex, say r, among T's leaves. We perform a depth-first search from r in order to obtain a directed spanning tree such that all tree edges are directed toward r (see Fig. 6-(c)). Note that every vertex of T, except for r, now has outdegree exactly one. (r has in-degree one and out-degree zero.) Furthermore, we find an undirected edge $(r, s) \in E$ such that $(r, s) \notin E_T$. By adding directed edge (r, s) to E_T, we obtain E_D (and hence we obtain $D = (V, E_D)$).

Lemma 1. *G is Hamiltonian if and only if G_a is Hamiltonian.*

Proof. Recall that exactly one vertex of each pair of multi-edges of $G_1 = (V_1, E_1)$ is in S. In other words, exactly one vertex of each pair of multi-edges of G_1 is *not* in S. Let $\overline{S} = V_1 - S$. Then, by removing all the vertices in $\overline{S}$ from G_a, we can decompose G_a into connected components, each of which corresponds to a vertex of G. Let CP_v denote the connected component corresponding to vertex v of G (see Fig. 8. v_1, v_2, and v_3 belong to $\overline{S}$.) Note that CP_v is a subgraph of G_a. Let x, y, and z be the three neighbors of v in G (see Fig. 3-(a)). (Intuitively, v_1, v_2, and v_3 in Fig. 8 correspond to the three edges (v, x), (v, y), and (v, z) in Fig. 3-(a), respectively.)

($\Leftarrow$) Suppose that there is a Hamiltonian circuit hc_a in G_a. Recall that v, x, y, and z of G correspond to subgraphs CP_v, CP_x, CP_y, and CP_z in G_a, respectively. We first construct a circuit c in G such that c passes through v from u to w if and only if hc_a passes through CP_v from CP_u to CP_w, where $u, w \in \{x, y, z\}$ and $u \neq w$. Since hc_a passes through every subgraph CP_v, the corresponding circuit c also passes through every vertex v of G. If hc_a passes through every CP_v *at most once*, then c passes through every vertex v of G at most once (and hence c is a Hamiltonian circuit of G). In the following, we show that hc_a passes through every CP_v at most once. Consider a subgraph CP_v of G_a. Suppose without loss of generality that hc_a passes through CP_v from CP_x to CP_y. Assume for contradiction that hc_a revisits CP_v. Since hc_a passes through CP_v from CP_x to CP_y, hc_a must revisit CP_v from CP_z. Since hc_a is a circuit, hc_a must pass through CP_v from CP_z to either CP_x or CP_y. If it is from CP_z to CP_x (resp. from CP_z to CP_y), hc_a must revisit the vertex in $\overline{S}$ between CP_v and CP_x (resp. between CP_v and CP_y), which contradicts the assumption that hc_a is a Hamiltonian circuit.

($\Rightarrow$) Let v_1, v_2, and v_3 be three vertices in $\overline{S}$ which correspond to edges $(v, x), (v, y)$, and (v, z) in G, respectively (see Fig. 8). It is not difficult to check that there is a Hamiltonian *path* in each CP_v between every two of the three vertices, v_1, v_2, and v_3 (see Fig. 9, symmetric cases are omitted). Suppose that there is a Hamiltonian circuit hc in G. We can construct the corresponding circuit c_a in G_a such that hc passes through v from u to w if and only if c_a passes through CP_v from CP_u to CP_w, where $u, w \in \{x, y, z\}$ and $u \neq w$. Since hc passes through every vertex v of G exactly once, c_a passes through every subgraph CP_v exactly once. The Hamiltonian path which passes through CP_v visits every vertex of CP_v's circles and triangle exactly once. Therefore, circuit c_a passes through every vertex of G_a exactly once. □

Lemma 2. *G_a is a Jordan-curve arrangement graph.*

Proof. Since G_a was constructed by adding circles to G_1, G_a is a Jordan-curve arrangement graph if G_1 is a Jordan-curve arrangement graph. In the following, we show G_1 is a Jordan-curve arrangement graph. Recall that the planar graphs G and G_1 are embedded in the plane so that none of their edges cross. Consider the following edge-coloring algorithm:

(1) Initially, all edges have no colors.

(2) Choose an arbitrary edge with no color, and color it with a new color.

(3) Find an edge, say e_1, with no color which satisfies the following condition: There exist three edges e_2, e_3 and e_4 such that e_1, e_2, e_3, and e_4 are incident to a vertex in clockwise order and that e_3 has already been colored. (See Fig. 4-(a).)

(4) Color e_1 with the same color as e_3.

(5) Repeat (3) and (4) until there is no edge e_1 satisfying the above condition.

(6) Repeat (2)-(5) until all edges are colored.

By applying this algorithm to G_1, we obtain subgraphs each of which consists of edges having the same color. Since G_1 is embedded in the plane so that none of its edges cross, these subgraphs are also embedded in the plane without any crossing edges. In the following, we show that each of these subgraphs is a 2-regular graph, i.e., it does not contain vertices of degree 4. (Note that if they are 2-regular graphs, we can construct the corresponding arrangement by replacing each 2-regular subgraph with a Jordan curve.)

Consider an arbitrary face f of G (see Fig. 10-(a)). Let $(a,b),(b,c)$, and (c,d) be edges on the boundary of f. By the transformation from G to G_1, (i) vertex b (resp. c) of G, which is the boundary point of three faces, is replaced by three edges, one of which is (b_a, b_c) (resp. (c_b, c_d)), and (ii) edge (b,c) of G, which is the boundary of two faces, is replaced by a pair of multi-edges between b_c and c_b (see Fig. 10-(b)). By the coloring algorithm, edge (b_a, b_c) is colored by the same color as one of the multi-edges between b_c and c_b, which is furthermore colored by the same color as edge (c_b, c_d). Thus, edges (b_a, b_c) and (c_b, c_d) are colored by the same color. From this observation, one can see that there exists a face f_a of G_a which corresponds to the face f of G such that edges (b_a, b_c) and (c_b, c_d) are on the boundary of f_a if vertices a, b, c, and d are on the boundary of f. More generally, exactly half of edges colored by the same color are on the boundary of a single face of G_a, and exactly one of every two adjacent edges colored by the color is on the boundary of the face. Therefore, edges colored by the same color form a 2-regular subgraph which corresponds to a face of G. □

Example. We give an example of arrangements of Jordan curves whose graphs are not Hamiltonian (see Fig. 7-(c)). The 3-regular planar graph shown in Fig. 6-(a) is not Hamiltonian, since it is not 1-tough [3], i.e., we can decompose it into *three* connected components by removing *two* vertices. This non-Hamiltonian graph can be reduced to the 4-regular planar graph with no multi-edges shown in Fig. 7-(b). This 4-regular graph is also non-Hamiltonian, since removing two subgraphs which correspond to the above two vertices decomposes the 4-regular graph into three connected components. (Although there is a Hamiltonian path in each of the three connected components, no Hamiltonian *circuit* can be constructed by connecting those three Hamiltonian paths.) Therefore, the arrangement of Jordan curves shown in Fig. 7-(c) is non-Hamiltonian.

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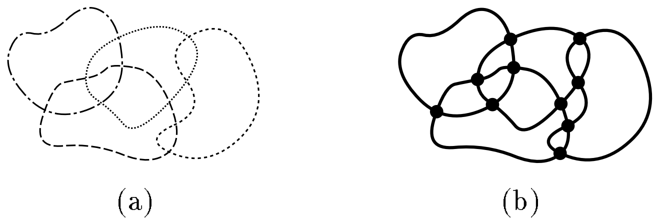


Fig. 1 (a) Arrangement of four Jordan curves (b) Jordan-curve arrangement graph

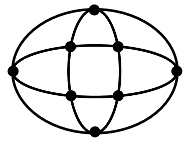


Fig. 2 A graph that cannot be formed by any arrangement of Jordan curves

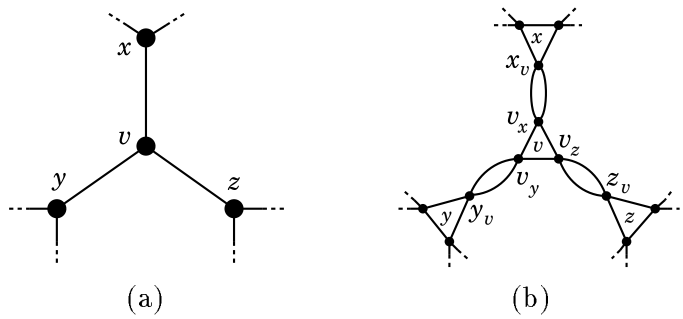


Fig. 3 (a) Vertices of G (b) Triangles of G_1

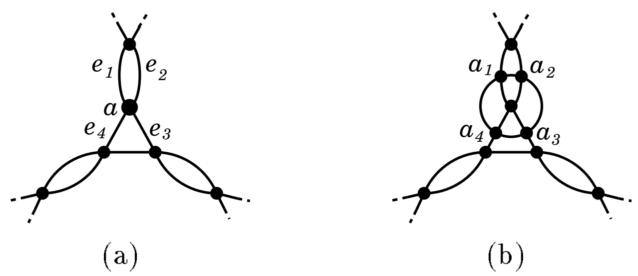


Fig. 4 (a) Four edges incident to vertex a in G_1 (b) Circle in G_a

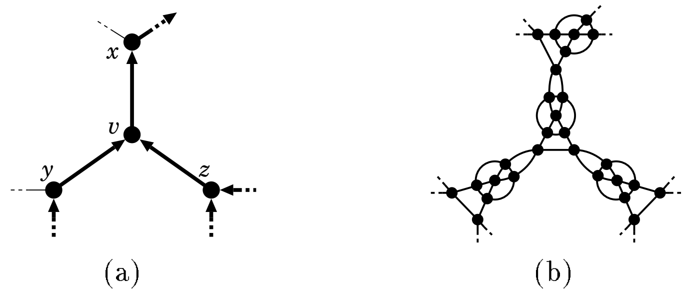


Fig. 5 (a) Directed subgraph D in G (b) Circles of G_a

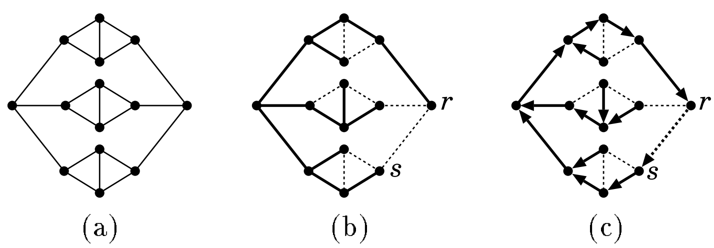
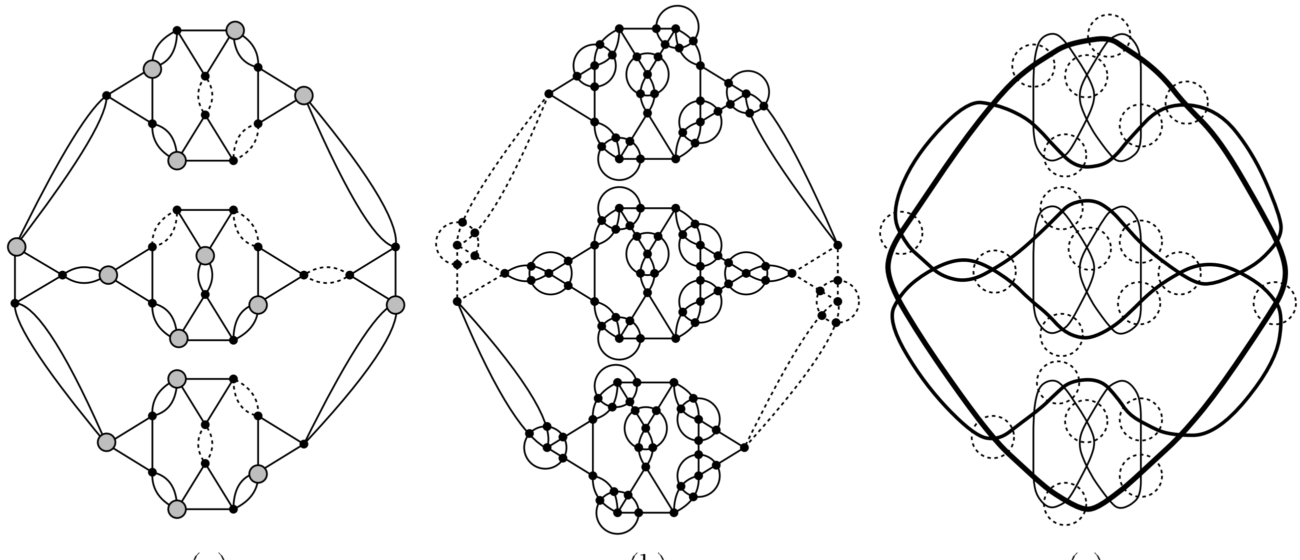


Fig. 6 (a) 3-regular graph (b) Spanning tree T (c) Directed subgraph D



(a) (b) (c)

Fig. 7 (a) Vertex set S_1 in G_1 (b) 4-regular planar graph G_a with no multi-edges
(c) Arrangement of Jordan curves

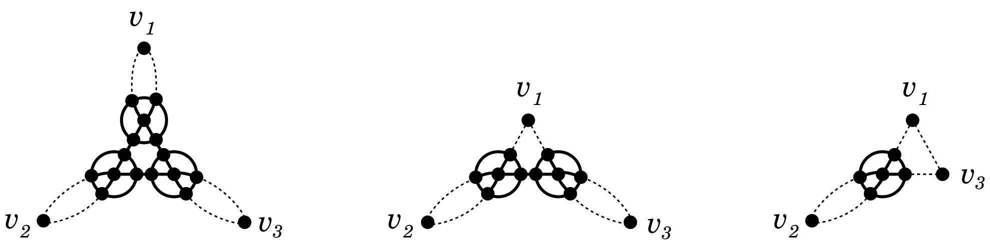


Fig. 8 Connected component CP_v

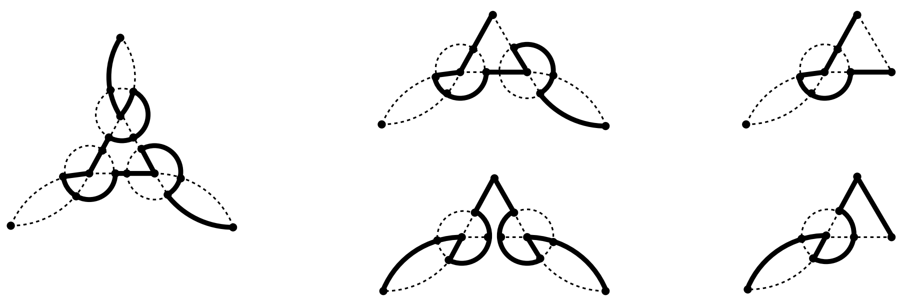
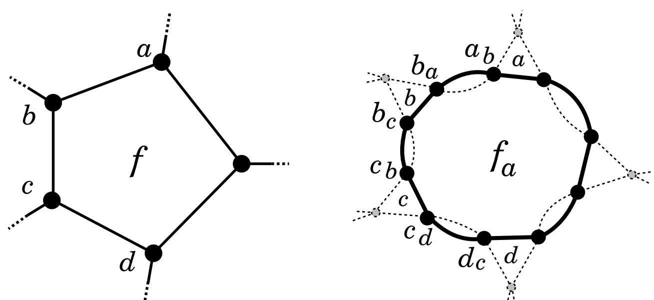


Fig. 9 Hamiltonian paths



(a) (b)

Fig. 10 (a) Face of G (b) Corresponding 2-regular subgraph in G_1