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chy of Figure 7. Recall from Figure 14 that three *monotone* polygons can *sequentially interlock* and four can interlock under simultaneous general motions. It is conjectured that three *unimodal* polygons are *sequentially separable* and four are always separable under simultaneous general motions.

Consider the problem of detecting and computing a translation ordering in a specified direction θ of a given set of polygons IP = $\{P_1, P_2, ..., P_M\}$, each of which contains n vertices. It is shown in theorems 4.10 and 4.11 that the detection and computation problems, respectively, can be solved in $O(\min(M^2n, Mn \log Mn))$ and $O(\min(M^3n, M^2n \log Mn))$ time. These upper bounds are obtained by a simple combination of existing algorithms. Thus it is reasonable to expect that these bounds can be improved.

Consider the problem of translating circles and spheres. In reference [20] it is shown that given a collection of circles $C = \{C_1, C_2, ..., C_n\}$, all the circles in C that can be translated to infinity individually without disturbing the others can be identified by taking each circle C_i in turn and applying the circle-reachability algorithm described therein. This produces O(n) sets of circles for which the Laguerre Voronoi diagram is computed O(n) times leading to the $O(n^2 \log n)$ complexity. Can this complexity be reduced? Perhaps this problem can be solved in O(n log n) time if the Laguerre Voronoi diagram [50] is computed only once on C and thereafter each circle is treated in O(log n) time in a *query* type mode. Another open problem is of course determining all the spheres that can be so translated in a collection of spheres $S = \{S_1, S_2, ..., S_n\}$ in 3 dimensions. There is no conceptual difficulty in applying the two-dimensional technique to this case. In fact, fast hidden sphere algorithms for intersecting spheres in 3 dimensions exists [51]. However, it is an open question how fast we can compute the Laguerre Voronoi diagram in 3 dimensions, or otherwise compute the *contour-surface* of the union of n spheres.

Theorem 5.7 states that two polyhedra P and Q *strongly monotonic* with respect to $PL(l_1)$ and $PL(l_2)$, respectively, are separable with a single translation when $l_1 = l_2$. It is an open problem whether this is true for $l_1 \neq l_2$. More generally, Dawson [27] has shown that any finite collection of *star-shaped* polyhedra are separable under simultaneous translations. It is not known whether this also holds true for *strongly monotonic* polyhedra in 3-space.

Section 5.3 illustrates some solutions to problems concerning the passing of a convex polyhedron P through a convex hole W in a “thin” wall. Theorem 5.5 gives a solution to the problem of determining whether P can be passed through W with a single translation and *no* pre-positioning. It is an open problem how fast this can be determined if *one translation* is also allowed for pre-positioning.

Finally, it is not yet known how fast we can determine if a convex polyhedron can be passed through a convex hole when arbitrary motions are allowed.

7. References

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Figure 20 illustrates two polyhedra P, Q *strongly monotonic* with respect to PL(z). It is taken from [44] and is known in Japanese carpentry as the “Ari-Kake” joint. Note that because of the “dove-tail” P and Q can only be separated with a translation in the z direction. It is proven in [39] that such polyhedra are always separable if they share a common direction *l*.

Theorem 5.7: Two polyhedra P, Q both of which are *strongly monotonic* with respect to PL(*l*) are separable with a single translation in direction *l*.

.5.6: Satin and Twills: The Computational Geometry of Weaving

Another class of movable separability problems occurs in the design of fabrics, i.e., weaving [47]. We give a brief description of one such problem but for details the reader is referred to [47] - [49]. We borrow some definitions from Grünbaum and Shephard [47].

The word *fabric* will be used in a mathematical sense to mean, roughly speaking, two layers of congruent strands in the same plane E such that the strands of different layers are nonparallel and they “weave” over and under each other in such a way that the fabric “hangs together”. To be precise, “weaving” means that at any point P of E which does not lie on the boundary of a strand, the two strands containing P have a stated *ranking*, that is to say, one strand is taken to have precedence over the other, and this ranking is the same for each point P contained in both strands. This concept may be conveniently expressed by saying that one strand *passes over* the other, in accordance with the obvious practical interpretation. By saying that the fabric *hangs together* we mean that it is impossible to partition the set of all strands into two nonempty subsets so that each strand of the first subset passes over every strand of the second subset.

It is convenient to let the plan E be the xy plane in 3-space and to consider translations of the strands in the z direction. Note that it is not necessary that the strands be ideal in the sense of having zero or negligible thickness. See, for instance, [46] for examples of “polyhedral fabrics” such as tabbies and twills. Here the *strands* are *convex weakly-monotonic* polyhedra. If the set of all *strands* admits a translation ordering in the z direction the fabric “falls completely apart”. Thus is our terminology determining whether a fabric *hangs together* is equivalent to determining if there exists a translation ordering of strict subsets of strands in the z direction.

A fabric can be represented by a matrix of 1’s and 0’s and for this representation efficient algorithms for determining whether a fabric hangs together are given in [48] and [49].

If we are given a “polyhedral fabric” we can first obtain the matrix representation of the fabric by solving the hidden surface problem for all strands taken together in the z direction.

6. Conclusion

We conclude by discussing some open problems in this area. It was mentioned in section 3 that a score of different families of polygons have made their appearance in the computational geometry literature. As we see in section 4, movability properties are knows for a few classes of polygons; convex, star-shaped, and a monotone being noteworthy examples. Consider the case of *unimodal* polygons. One would expect unimodal polygons to experience a greater degree of freedom of motion than monotone polygons since unimodal polygons are closer to rectangles in the hierar-

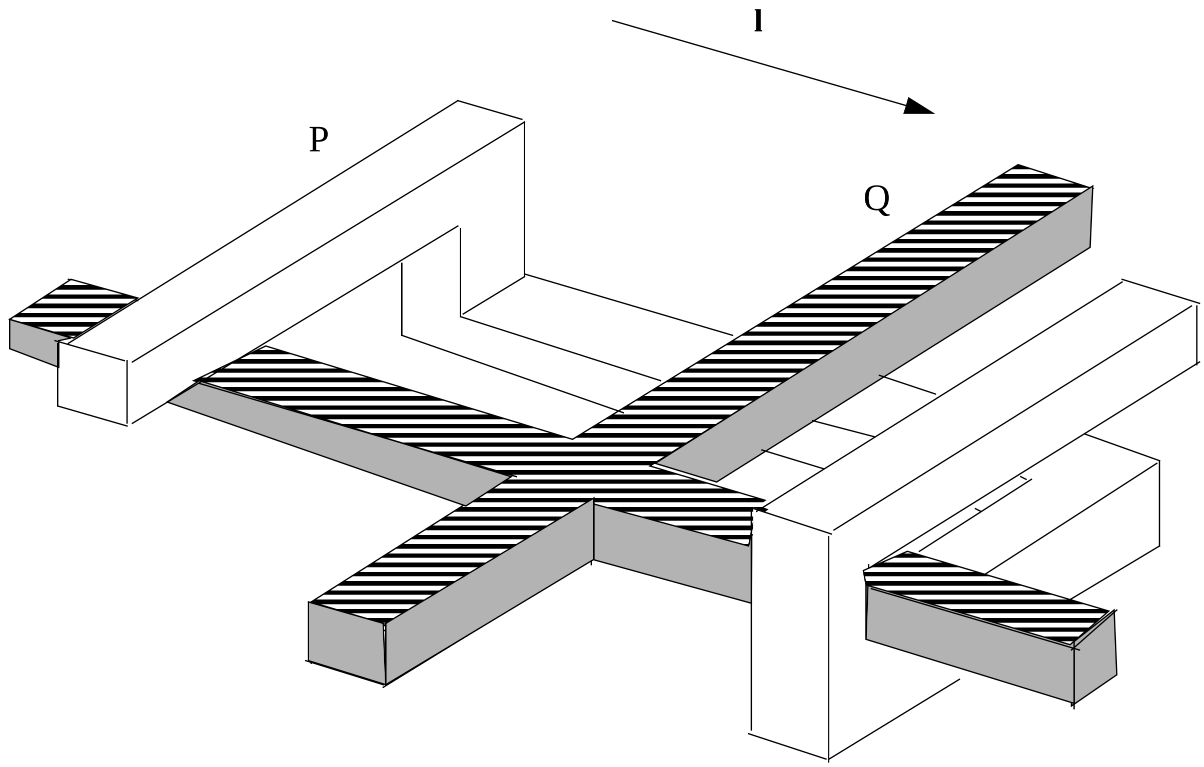


Figure 19: Two polyhedra weakly monotonic with respect to a common direction can interlock under all motions.

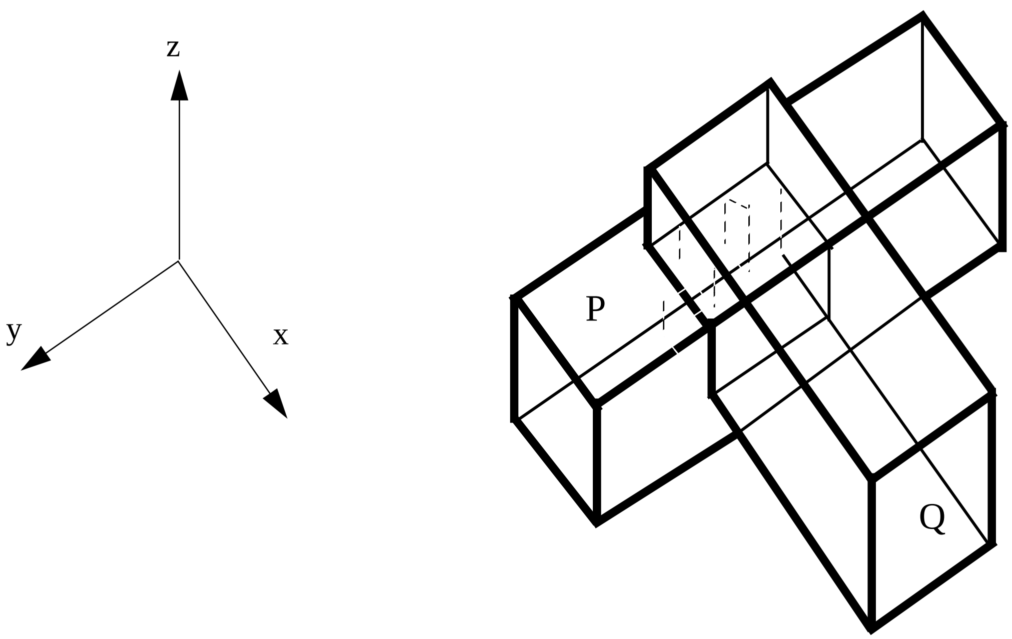


Figure 20: Two polyhedra strongly monotonic with respect to PL(z). This example is take from[44] and illustrates the “Ari-Kake” joint in Japanese carpentry. The only way to separate this pair of polyhedra is to translate either P or Q in the z direction.

Fact: Two polyhedra P, Q *weakly monotonic* with respect to a common direction *l* can *interlock* under *all motions*.

Consider the two polyhedra in Figure 19. P is an “isothetic coil” and Q an “isothetic cross”. They are weakly monotonic with respect to a direction parallel to the x axis. By thickening the four “arms” of the “cross” until they touch the “coil” we cannot subsequently move one without the other. This holds true for all possible *translations*, *rotations*, and *screw* motions, i.e., simultaneous translations and rotations.

Another characterization of monotone polygons in the plane follows from the definition of *directional convexity*. A polygon P is monotonic in direction *l* if for every pair of points a, b $\in$ P, such that the line L(a,b) through a and b is orthogonal to *l*, the *line segment* [a,b] lies in P. Alternately, we say the polygon is *directionally convex* with respect to $l + \pi/2$. Several possibilities exist for generalizing this notion to higher dimensions.

Definition: A polyhedron P is *directionally convex* with respect to *l*, if there exists a direction *l* such that for every pair of points a, b $\in$ P with L(a,b) *parallel* to *l*, the line segment [a,b] lies in P.

Fact: Two polyhedra P, Q *directionally convex* with respect to a common direction *l* can *interlock* under *all motions*.

The example of Figure 19 illustrates the point. Both P and Q are directionally convex with respect to the y axis.

Note that if a polyhedron P is directionally convex with respect to *l* it does not necessarily follow that P is weakly monotonic with respect to all directions orthogonal to *l*. Polyhedron P in Figure 19 is directionally convex with respect to the y axis but it is not weakly monotonic in the z direction. In the above definition only one direction *l* was used for convexity. We can obtain different families of directionally convex polyhedra by increasing the number of directions used. One way of doing this is to consider all directions lying on a plane, as in the following definition.

Definition: A polyhedron P is *directionally convex orthogonal* to *l*, if there exists a direction *l* such that for every pair of points a, b $\in$ P lying on a *plane orthogonal* to *l*, the line segment [a,b] lies in P.

Note that such a polyhedron must be *weakly monotonic* with respect to *l* in the *convex* sense.

Under all three definitions above we have seen that two polyhedra can interlock under all motions. Now we introduce a class of polyhedra which we call *strongly monotonic* which possesses some movability properties analogous to monotone polygons in the plane. Let PL(*l*) denote a plane orthogonal to a direction *l*.

Definition: A polyhedron P is *strongly monotonic with respect to* PL(*l*) if there exists a direction *l* such that all planes *parallel* to *l* that intersect P form as their intersection with P a *simple* polygon that is *monotonic* in direction *orthogonal* to *l*.

the cones in D gives another cone which is the set of directions for simultaneous translation of all the p_i, and hence of P. Each cone can be computed in O(m) time and thus all cones can be found in O(mn) time. All the cones can be translated to D in O(mn) time. All that remains is to compute the intersection of all the cones. Now, each cone can be viewed as the intersection of m half-spaces determined by the planes co-planar with the m faces of each cone. The *interior* half-space contains the cone. Therefore the solution cone is the intersection of all the interior half-spaces determined by all the cones in D. Now, the intersection of k half-spaces in 3-dimensional space can be computed in O(k log k) time using an algorithm of Preparata and Muller [45]. Therefore the solution cone can be obtained in O(mn log mn) time. Q.E.D.

5.4: Separating star-shaped polyhedra

The results of section 4.3 on star-shaped polygons in the plane extend to three dimensions. Let $\mathbf{P} = \{P_1, P_2,..., P_M\}$ be a collection of M non-intersecting star-shaped polyhedra with n vertices each. Dawson [27] has shown that **P** can always be separated with simultaneous translations. Let K_i be the *kernel* of P_i and let k_i be a point in K_i. Let x be any point in 3-space. The vector $\overline{xk_i}$ determines the velocity and direction of translation for P_i in a valid set of simultaneous translations. Since the linear programming algorithm of Dyer [24] runs in linear time in 3 dimensions as well, we have the following theorem.

Theorem 5.6: Let **P** be a collection of star-shaped polyhedra. A set of translations for simultaneous separation of **P** can be determined in O(nM) time.

.5.5: Interlocking monotone polyhedra

In the previous section we observed that some of the movability properties of star-shaped polygons in the plane carry over exactly to star-shaped polyhedra in 3-space. The definition of monotone polygons, on the other hand, does not generalize straightforwardly or uniquely to three dimensions. In this section we explore several families of “monotone” polyhedra and consider some of their separability properties.

One common definition or characterization of monotone polygons is as follows. A polygon P is monotonic in direction *l* if for every line L orthogonal to *l* that intersects P, the intersection $L \cap P$ is a line segment (or point). We generalize this definition to 3-dimensional space to obtain a family of monotone polyhedra we call *weakly monotonic*.

Definition: A polyhedron P is *weakly monotonic* in direction *l* if there exists a direction *l* such that each plane orthogonal to *l* that intersects P, yields a *simple polygon* (or a line segment or point).

Note that there exists a score of rather well-known special classes of simple polygons [17], [42] - [43]. By substituting these for the word *simple* in the above definition we obtain a score of families of *weakly monotonic* polyhedra. Thus we say that if all the intersections are *convex*, we have a *weakly monotonic* polyhedron in the *convex sense*. Figure 18 illustrates a *weakly monotonic* polyhedron in the *monotonic* sense, i.e., with monotonic polygons as intersections.

In [35] it was shown that two monotonic polygons in the plane, even if they do *not* share a *common* direction of monotonicity, can always be separated with a single translation. In three dimensions, on the other hand, *weakly monotonic* polyhedra lose all their freedom of motion, even if they share a common direction of monotonicity, as we now demonstrate with an example.

Theorem 5.3: Given a convex polyhedron $P = (p_1, p_2,..., p_n)$ and a convex window $W = (w_1, w_2,..., w_m)$ on a plane H, whether P can pass through W by a *single translation* orthogonal to H, after prepositioning with only a single translation, can be determined in O(n + m) time.

Note that if SH(P, θ), where θ is orthogonal to H, cannot be contained in W it is still possible that P can pass through W by a single translation after pre-positioning with only a single translation.

Chazelle [41] has also shown that given two convex polygons Q(n) and R(m) whether Q can be contained in R by translations and rotations can be determined in $O(nm^2)$ time, and we therefore have.

Theorem 5.4: Given a convex polyhedron $P = (p_1, p_2,..., p_n)$ and a convex window $W = (w_1, w_2,..., w_m)$ on a plane H, whether P can pass through W by a *single translation* orthogonal to H, after pre-positioning with arbitrary translations but rotations only with respect to axes of rotation orthogonal to H, can be determined in $O(nm^2)$ time.

A related problem asks for the smallest, in some sense, window W on a plane H through which a given convex polyhedron P can be passed with a single translation orthogonal to H after pre-positioning P with arbitrary translations and rotations. This problem is directly related to finding the smallest shadows. McKenna and Seidel [37] give an $O(n^2)$ algorithm for computing the minimum-area shadow of a convex polyhedron with n vertices.

As a final example of this type of problem, we can ask whether there exists a direction for a polyhedron to pass through a hole by a single translation *without* pre-positioning. Let $P = (p_1, p_2,..., p_n)$ be a convex polyhedron arbitrarily positioned, on one side of a plane H, with respect to a convex window $W = (w_1, w_2,..., w_m)$ in H. Then we obtain the following lemma.

Lemma 5.3: P can pass through W with a single translation in direction θ if, and only if, each vertex of P can be passed through W with a single translation in direction θ.

Note that this no longer holds true for non-convex holes in H.

Lemma 5.3 allows us to solve the above problem. In fact we can do more; we can compute *all* directions for passing P through W with a single translation and *no* pre-positioning. Alternately, we can view this problem as computing *all* the shadows SH(P, θ) on H that can be contained in W.

Theorem 5.5: Given a convex polyhedron $P = (p_1, p_2,..., p_n)$ arbitrarily positioned on one side of a plane H, with respect to a convex window $W = (w_1, w_2,..., w_m)$ in H, all directions for translating P through W can be computed in O(mn log mn) time.

Proof: By lemma 5.3 we can restrict ourselves to computing all directions for simultaneously translating the *vertices* of P through W. Consider vertex p_i. All directions for translating p_i from its initial configuration through W are defined by all the vectors emanating from p_i and intersecting H in W. Therefore the *cone* determined by the half-lines from p_i through w_j, j = 1,...,m specifies all such directions for p_i. Denote such a cone by CONE(p_i, W). Construct a 3-dimensional euclidean *direction* space D and translate all the cones CONE(p_i, W), i = 1, 2,..., m in D such that the p_i all overlap with the origin of D. Then the intersection of all

Dawson [27] has shown that given a collection of n spheres in 3-space at least min{n, 4} of them can be translated to infinity without disturbing the others. Lemmas 5.1 and 5.2 allow us to sharpen this result.

Let CH(O) denote the convex hull of O = $\{o_1, o_2, ..., o_n\}$ and let H be the number of sphere centers that lie on vertices, edges, and faces of CH(O). We then have the following theorems [20].

Theorem 5.2: Given $S = \{S_1, S_2, ..., S_n\}$, then the number of spheres that can be translated to infinity without disturbing the others is at least H and they can be identified in O(n log n) time.

5.2: Translating convex polyhedra

Consider first the *isothetic* case. It turns out that four isothetic rectangular polyhedra can be arranged such that for some directions in 3-space no translation ordering exists. The example due to Guibas and Yao [8] is reconstructed in Figure 17. A more surprising result is that if we relax the *isothetic* requirement we can find sets of rectangular polyhedra that do not admit a translation ordering in *any* direction. Therefore, convex polyhedra in 3-space do *not* exhibit the translation *ordering property*. An example of twelve convex polyhedra that do not admit an ordering in any direction is given in [39]. In the example in [39], two polyhedra can be moved to infinity for some chosen directions. An even more surprising result due to Dawson [27] is an example of twelve convex polyhedra none of which may be translated in any direction without disturbing the others. A similar example with only six objects was discovered by Post [40].

5.3: Passing a convex polyhedra through a window

A generalization of the problem illustrated in Figure 8 asks for whether a *convex* polyhedron P can be *passed* through a *convex window* W. Here W may be a circle, a rectangle, or an arbitrary convex polygon. Let H be a plane not intersecting P and assume a light source at infinity somewhere on the same side of H as P.

Definition: The *shadow* of P, denoted by SH(P), is a convex polygon determined by the projection of P onto H, i.e., the portion of H not illuminated. If θ is the direction of the "light rays" we will also use SH(P, θ) to denote the shadow of P in direction θ.

One would hope for a theorem analogous to theorem 4.1 relating SH(P) to the window W. Unfortunately this is not the case. While it is true that if there exists a direction of projection θ such that SH(P) fits into W, then P can be passed through W (a single translation will do), there exists cases where no shadow of P on any plane H fits into W and yet P can still be passed through W by some sequence of displacements [21]. Nevertheless, the *shadow* concept is still relevant to the problem of separability, in particular if we are interested in separation with a single translation after initial positioning. We can also limit the initial positioning displacements to translations only. Let $P = (p_1, p_2, ..., p_n)$ be a convex polyhedron and let $W = (w_1, w_2, ..., w_m)$ be the convex window on a plane H. Chazelle [41] has shown that given two convex polygons with n and m vertices, respectively, whether the first can be fitted into the second with translations only can be determined in O(n + m) time. Furthermore, given a polyhedron P in 3-space and a direction θ orthogonal to a plane H, the SH(P, θ) can easily be computed in O(n) time. We thus have the following theorem.

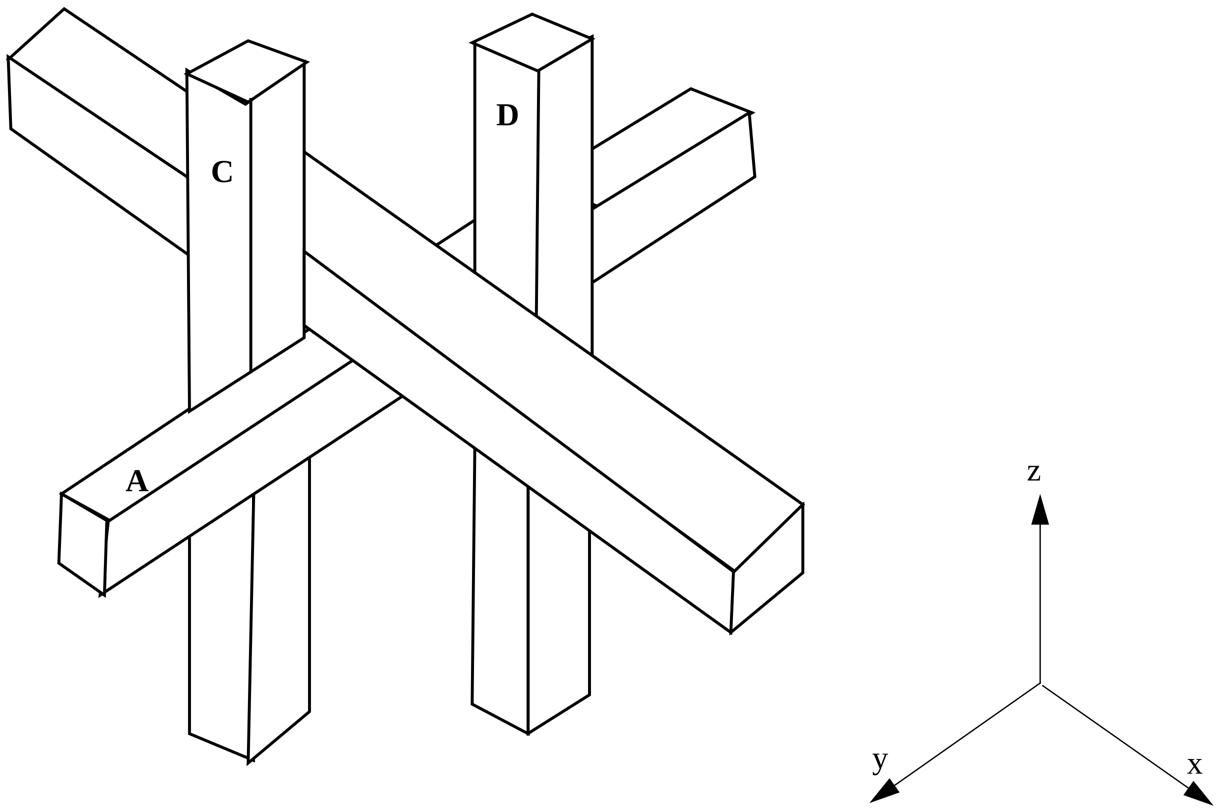


Figure 17: A set of isothetic convex polyhedra that does not allow translation ordering in some directions such as x + y.

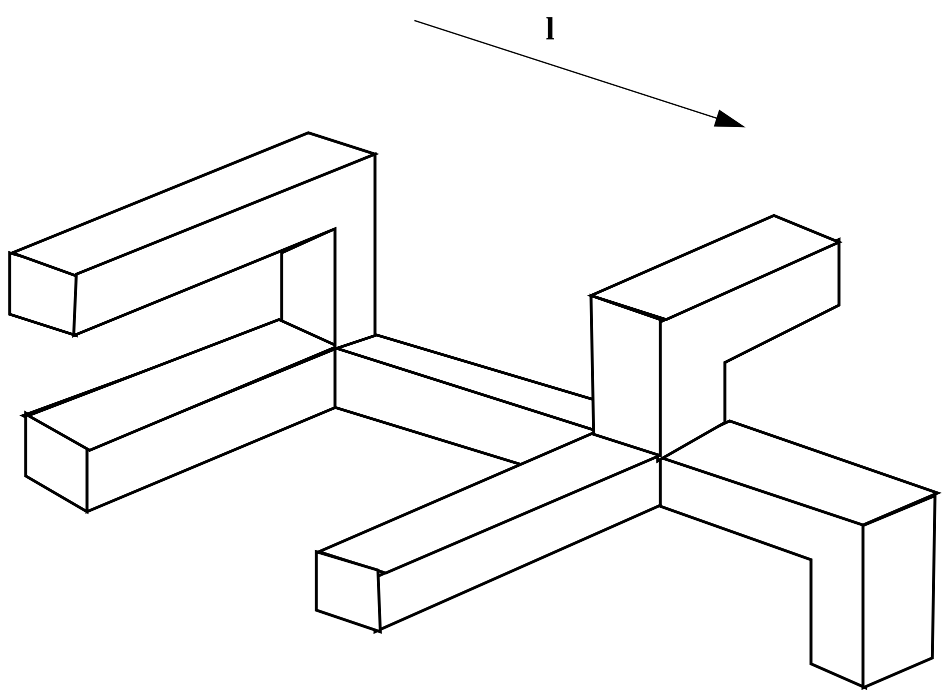


Figure 18: Illustrates a weakly monotonic polyhedron with monotonic intersections.

idea with the following example. Recall from Figure 14 that three star-shaped polygons can be *sequentially interlocked*. If on the other hand the polygons are *isothetic* then any number of them are sequentially separable.

Theorem 4.12: A collection of M *isothetic star-shaped* polygons admits a *translation ordering* in the x and y directions.

Proof: First we note that if an isothetic polygon is *star-shaped* then it is *monotonic* in the x and y directions [38]. The result then follows from lemma 4.1 and theorem 4.9. Q.E.D.

5. Separability in Three Dimensions

5.1: Translating spheres

In section 4.1 it was pointed out that the *line-sweep* heuristic is successful in determining a translation ordering in the x and y directions for a collection of *isothetic* rectangles. It is clear that the arguments carry over to three dimensions (x, y, z directions) for *isothetic* rectangular solids. It turns out that it also works correctly for sets of spheres of equal radii [20]. Furthermore, a variation of *line sweep* will work for unequal spheres [20]. (Actually, a *plane-sweep* in 3-D)

Let $S = \{S_1, S_2, ..., S_n\}$ denote a set of n non-intersecting spheres in 3-space with centers $O = \{o_1, o_2, ..., o_n\}$, where S_i is specified by the cartesian coordinates of o_i, namely (x_i, y_i, z_i), and the radius r_i. A *great circle* on a sphere S_i is a circle on the surface of S_i that partitions S_i into two *hemispheres*. Consider a sphere S_i and assume there is a light source at infinity casting rays of light in direction l. The *shadow tunnel* of S_i in direction *l*, denoted by ST(S_i, *l*) is that subset of space not illuminated by the light source along with its boundary.

Lemma 5.1: Given $S = \{S_1, S_2, ..., S_n\}$, S_i can be translated to infinity in direction *l* without disturbing the other spheres if, and only if, no point of S_j, j = 1, 2,..., n, j ≠ i lies in the interior of ST(S_i, *l*).

Let HS(S_i^+, *l*) denote the open half-space determined by the plane orthogonal to *l* that cuts S_i at a *great circle* and contains ST(S_i, *l*). Similarly let HS(S_i^-, *l*) denote the complement of the union of HS(S_i^+, *l*) and the cutting plane. These half-spaces will be referred to as closed when the cutting plane is included in the set.

Lemma 5.2: Given $S = \{S_1, S_2, ..., S_n\}$, if there exists a direction *l* and a sphere S_i such that the o_j, j = 1, 2,..., n, j ≠ i all lie in closed HS(S_i^+, *l*) then S_i can be translated to infinity in direction *l* without disturbing the other spheres.

Lemmas 5.1 and 5.2 imply that if we apply *plane-sweep* to the *centers* of the spheres rather than the spheres themselves we can obtain a valid translation ordering. We thus obtain the following two theorems proved in [20].

Theorem 5.1: Given $S = \{S_1, S_2, ..., S_n\}$, for *all* directions *l* there exists an ordering on *S* such that the spheres can be translated by some common vector in direction *l one at a time* without collision and such an ordering can be computed in O(n log n) time.

Theorems 4.7 and 4.9, together with existing solutions to certain geometric problems [30] - [32], [36] offer immediate algorithms for solving the problems of *detecting* and *computing* translation orderings for sets of simple polygons.

The Detection Problem

Theorem 4.10: Given a collection of M simple n-gons **P** = $\{P_1, P_2, ..., P_M\}$ and a direction θ, whether or not a *translation ordering exists* can be determined in time $O(\min(M^2n, Mn \log Mn))$.

Proof: From theorems 4.7 and 4.9, it follows that we can first compute the visibility hulls $VH(P_i,\theta)$, i = 1, 2,..., M and subsequently determine if any pair intersects. The first step can be done in O(Mn) time [30] - [31]. The second step can be done in two ways. Since the $VH(P_i,\theta)$ are monotonic polygons with a common direction of monotonicity $\theta + \pi/2$, it follows that whether any pair intersects can be determined in O(n) time. Testing all such pairs yields $O(M^2n)$ time for this method. The second method is to consider the M n-gons as a set of Mn line segments and to use the method of Shamos & Hoey [32] to yield a complexity of O(Mn log Mn). Q.E.D.

Note that which method is faster depends on whether M or n dominates in a given problem. If M is a constant we have O(n) versus O(n log n); if n is a constant we have $O(M^2)$ versus O(M log M).

The Computation Problem

Theorem 4.11: Given a collection of M simple n-gons **P** = $\{P_1, P_2, ..., P_M\}$ and a direction θ that admits a translation ordering of **P**, then such a translation ordering can be computed in time $O(\min(M^3n, M^2n \log Mn))$.

Proof: Two approaches are immediately obvious.

Method 1: For every top vertex construct a half-line in direction θ and determine if it intersects any of the remaining Mn-1 edges. This can be done in O(M2n) time. Now in O(M) time we can determine which polygon is the first to move. We repeat this for the remaining polygons until none remain for a total complexity of $O(M^3n)$.

Method 2: Considering the M polygons as a set of Mn line segments, we can compute the *next-element subdivision* in direction θ in O(Mn log Mn) time [36]. From this structure we can pick out the first polygon to move in O(M) time. Repeating for the remaining polygons leads to a total complexity of $O(M^2n \log Mn)$. Q.E.D.

Note that, as before, which method is faster depends on whether M or n dominates. Hossam ElGindy has reduced the complexity of both the above problems to O(Mn log M).

4.6: Isothetic polygons

In some applications areas such as VLSI [8] the collection of polygons is *isothetic* with respect to a *common* pair of orthogonal direction (say the x and y axes). One would expect movability to enjoy some additional freedom as compared to the case of arbitrary polygons. We illustrate this

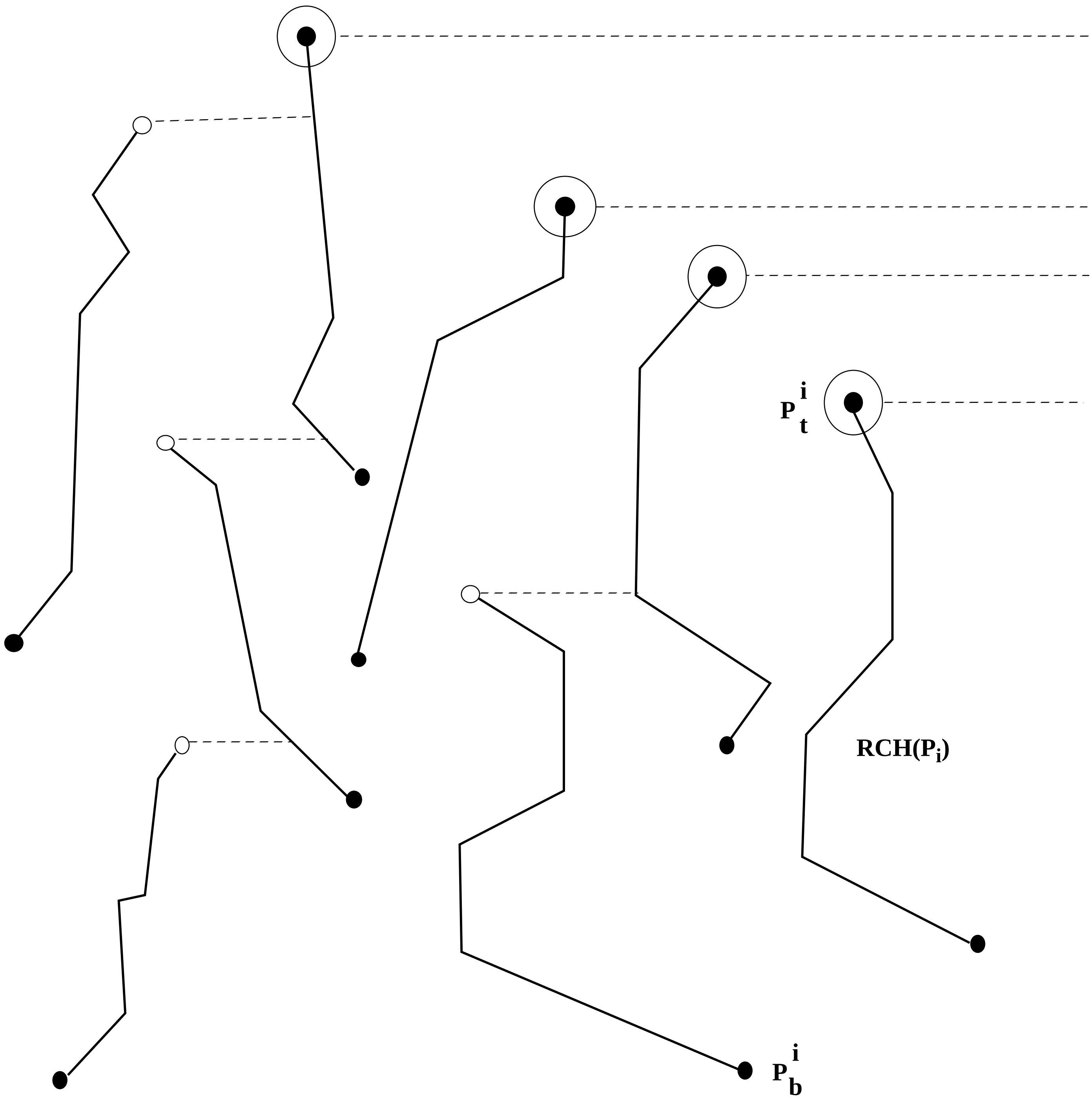


Figure 16: The monotone chain with the lowest illuminated top vertex can always be moved out first.

Theorem 4.8: Given two simple n-gons P and Q, and a direction θ, whether or not P and Q are *separable* by a single translation in direction θ can be determined in O(n) time.

Proof: From theorem 4.7 it follows that it is sufficient to compute the visibility hulls of P and Q in the direction θ and then determine if the hulls intersect. The first step can be done in O(n) time with either of the algorithms of El Gindy and Avis [30] or Lee [31]. The second step can be done in O(n) time with a slight variation of the "slab" method of Shamos & Hoey [32] for intersecting convex polygons. Q.E.D.

Theorem 4.9: A set of polygons IP = $\{P_1, P_2, ..., P_M\}$ admits a *translation ordering* in direction θ if, and only if, every pair of polygons, viewed in isolation, is separable with a single translation in direction θ.

Proof: Without loss of generality let θ be the x axis. First consider the case when a translation ordering exists. Let each polygon be translated in *order* by some fixed magnitude and assume we have just translated $P_{(i)}$ from position A to position B. Clearly $P_{(i)}$ is separable from each polygon in the set $\{P_{(1)}, P_{(2)}, ..., P_{(i-1)}\}$. Furthermore, $P_{(i)}$ can be translated back to A from B. Therefore $P_{(i)}$ is separable from each polygon in the set $\{P_{(i+1)}, P_{(i+2)}, ..., P_{(M)}\}$. Since this remains true for all i it follows that every pair of polygons is separable with a single translation in direction θ. Next, consider the case where every pair of polygons, viewed in isolation, is separable with a single translation in direction θ. We must show that **P** admits a translation in direction θ. From theorem 4.7 it follows that for all i and j $int[VH(P_i,\theta)] \cap int[VH(P_j,\theta)] = \phi$. Thus it is sufficient to show that if we are given M non-intersecting polygons monotonic in a common direction $\theta + \pi/2$, then they admit a translation ordering in direction θ. Let $RCH(P_i)$ denote the *right chain* of P_i, i.e., $RCH(P_i) = (p_t^i, p_{t+1}^l, ..., p_b^l)$ where p_t^i and p_b^l are the vertices with maximum (top) and minimum (bottom) y coordinates, respectively. Clearly, a polygon can be translated in direction θ if, and only if, its *right chain* can be so translated. Hence, we need only consider the right chains. Now imagine a light source at $x = +\infty$ and mark all the top vertices of the chains which are illuminated (see Figure 16). To establish that a translation ordering exists it is sufficient to show that one of these monotonic chains can always be translated to $x = +\infty$ without disturbing the others. It turns out that the chain with the lowest (minimum y coordinate) *marked top* vertex can always be moved out first. Let $RCH(P_i)$ be the desired chain and assume it cannot be translated to $x = +\infty$. This means that there must be another chain, say $RCH(P_j)$, blocking $RCH(P_i)$. Two cases arise depending on how $RCH(P_j)$ blocks $RCH(P_i)$.

Case 1: p_t^j lies above p_t^i. Because the monotonicity of the chains it that follows if $RCH(P_j)$ is to block $RCH(P_i)$ then p_t^icannot be illuminated, a contradiction.

Case2: p_t^j lies below p_t^i. Because of monotonicity it follows that if $RCH(P_j)$ is to block $RCH(P_i)$ then p_t^j must lie above p_b^i and to the right of $RCH(P_i)$. Now p_t^j cannot be illuminated since it is below p_t^i and would lead to a contradiction. Therefore p_t^j must be blocked by some third chain $RCH(P_k)$. Furthermore, using similar arguments to the above p_k^t must lie in between p_j^t and p_i^t. But this in turn requires an unlimited number of additional blocking chains. Since we only have M available we eventually obtain an illuminated vertex lower than p_t^i, a contradiction. Q.E.D.

Step 4: Compute VH(Q,$\theta + \pi/2$).

Step 5: If P intersects VH(Q,$\theta + \pi/2$)

then EXIT with $\psi \leftarrow \phi + \pi/2$

else EXIT with $\psi \leftarrow \theta + \pi/2$

End

Theorem 4.6: Algorithm SEPARATE determines a direction of separability for two monotone polygons P and Q in O(n) time.

Proof: The correctness of the algorithm follows from theorem 4.5. Thus we turn to the complexity. Steps 1-3 can be performed in O(n) time using the algorithm in [29] by Preparata and Supowit. Computing the visibility hull of Q in step 4 can be done in O(n) time with a variety of hidden line algorithms [30]-[31]. Finally, step 5 can be performed in O(n) time using a simple modification of the slab method of Shamos and Hoey [32] for intersecting two convex polygons. This follows from the fact that P and VH(Q,$\theta + \pi/2$) are two polygons monotonic in direction θ and therefore their intersection can only contain a linear number of pieces. See for example Guibas and Stolfi [33]. Q.E.D.

Just as *star-shaped* polygons can be sequentially interlocked but any number of them are always separable under simultaneous translations, so we can ask this question for monotone polygons. Dawson [34] has shown that three monotone polygons are separable with simultaneous translations but *four* can interlock under simultaneous general motions (see Figure 15).

4.5: Simple polygons

Since arbitrary simple polygons may or may not interlock in a variety of senses, it is interesting to determine if a configuration of polygons does or does not interlock in any of these senses. We illustrate some results along these lines.

Theorem 4.7: Two simple polygons P and Q are movably separable by a single translation in direction θ if, and only if, int[VH(P,θ) $\cap$ int[VH(Q,θ)] = ϕ where ϕ is the null set.

Proof: (sufficiency) In this case we have two polygons, the interiors of which do not intersect, and which are monotonic in a common direction $\theta + \pi/2$. Thus the result follows from the lemma 4.1. For necessity four cases suffice:

(a) A point $z \in$ int(Q) lies in the interior of a pocket of VH(P). In this case, if z is translated in direction θ it cannot exit the pocket through its lid since the lid is parallel to θ. Therefore z must collide with P and so must Q.

(b) A point $z \notin$ int(Q) but $z \in$ intVH[Q] lies in a pocket of VH(P). In this case if z is translated in direction θ it must collide either with P or Q. In the former sub-case we are done. In the latter sub-case let z collide with Q at $z' \in$ Q. But now case (a) applies with z'. The last two cases where $z \in$ int[VH(P)] are similar. Q.E.D.

Note that theorem 4.3 provides additional ways of separating the configuration of Figure 6 by *simultaneous* translations. Since $P_1 \cup P_2$ and $P_3 \cup P_4$ make up two star-shaped polygons that can be separated with one translation. Since all four polygons are themselves star-shaped two more translations will separate P_2 from P_1 and P_3 from P_4.

Theorem 4.3 also implies that two star-shaped polygons can be separated by translating *both* P and Q *simultaneously* in some pairs of directions with respect to an arbitrary fixed point in the plane. It is sufficient to guarantee that the *relative* motion between P and Q is correct. Accordingly, let a* and b* be any pair of points in the plane such that L(a*,b*) intersects K(P) and K(Q). Let x be any reference point on the plane, and consider the vectors $\overline{xa^*}$, $\overline{xb^*}$ and $\overline{a^*b^*}$. We can see now that if we translate P and Q in the directions of $\overline{xa^*}$ and $\overline{xb^*}$ with velocities proportional to the magnitudes of $\overline{xa^*}$ and $\overline{xb^*}$, respectively, the correct relative motion between P and Q is maintained. Different pairs of points a*, b* only change the relative velocity of separation. An alternate, very elegant, proof of this result for the restricted case in which $a^* \in K(P)$ and $b^* \in K(Q)$ was given by Dawson [27].

So much for *two* polygons - what about *three* star-shaped polygons. It turns out that as few as three star-shaped polygons can *sequentially interlock*. One such example is shown in Figure 14. A surprising result, however, due to Dawson [27] is that any finite collection of star-shaped polygons can still be separated by simultaneous translations.

4.4: Monotone polygons

Examination of Figure 14 reveals that the three polygons are monotonic in addition to being star-shaped, and thus three monotonic polygons can be sequentially interlocked. Furthermore form lemma 4.1 we know that two monotonic polygons are separable with the restriction that they share a common direction of monotonicity. This restriction is removed in [28], [35] with the following theorem.

Theorem 4.5: Given two polygons P and Q monotonic in direction θ and ϕ, respectively, then P and Q are separable with a single translation in at least one of the two directions $\theta + \pi/2$, $\phi + \pi/2$.

This theorem immediately suggests the following algorithm for determining a direction of separability for two monotone polygons.

<u>Algorithm SEPARATE</u>

Input: Two non-intersecting monotone polygons $P = (p_1, p_2,..., p_n)$ and $Q = (q_1, q_2,..., q_n)$.

Output: A direction ψ for separating P and Q.

Begin

Step 1: Compute the directions of monotonicity for P and Q.

Step 2: If P and Q have a common direction of monotonicity ζ,

then EXIT with $\psi \leftarrow \zeta + \pi/2$

Step 3: Pick two directions of monotonicity for P and Q, say θ and ϕ respectively.

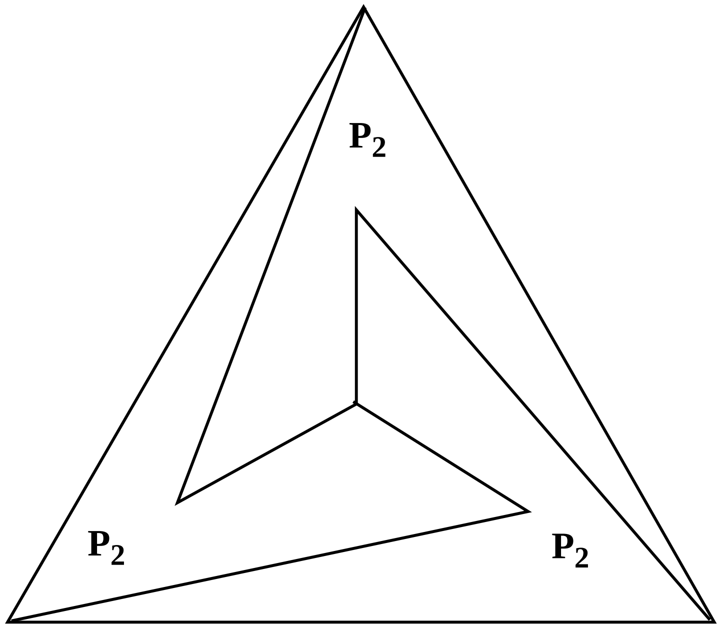


Figure 14: Three sequentially interlocking star-shaped polygons.

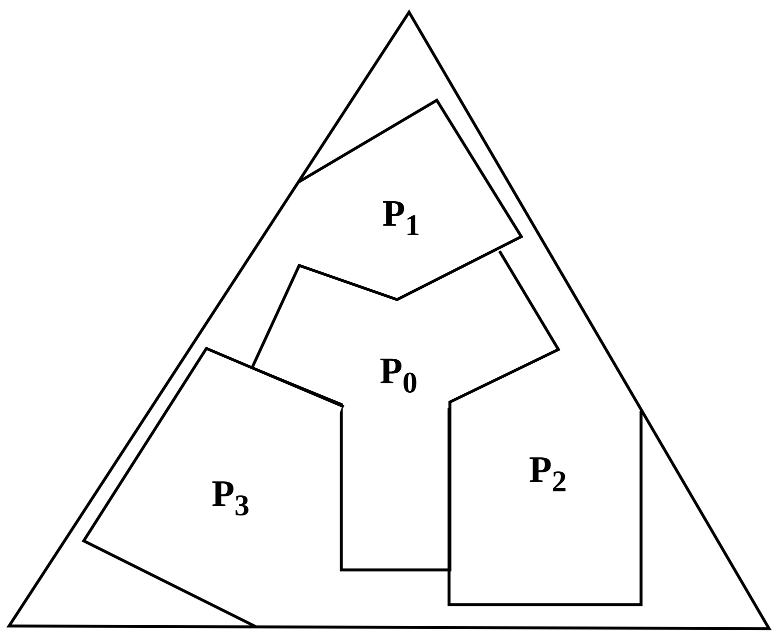


Figure 15: Four monotonic polygons interlocked under simultaneous general motions (From Dawson [34]).

The *visibility deficiency* polygons or “pockets” are of two types: *lower pockets* and *upper pockets*. Let p_i be a vertex of P that determines some pocket. If p_{i-1} and p_{i+1} do not lie above the line collinear with the lid of the pocket then the pocket is a *lower* pocket. Similarly if p_{i-1} and p_{i+1} do not lie below the lid line then we obtain an *upper* pocket. (We assume here without loss of generality that θ is in the direction of the x axis.)

Lemma 4.3: If L_z is a straight line in the direction θ through a point $z \in K(P)$ then int[VD(P,θ)] is *not visible* from any point on $L_z \cap$ ext[VH(P,θ)].

Proof: (Refer to Figure 12) First we note that a pocket whose lid is above L_z is a lower pocket. For if it were an upper pocket it would imply, by Jordan curve theorem that P was not star-shaped from z, a contradiction. Similarly, a pocket whose lid is below L_z must be an upper pocket. It follows that any point x in int(upper pocket) can only see points y lying below L_z. Similarly, any point x in int(lower pocket) can only see points y above L_z. Q.E.D.

Theorem 4.3: Two *star-shaped* polygons are movably separable with a single translation.

Proof: (Refer to Figure 13) Let P and Q be two non-intersecting star-shaped polygons. Let $a \in K(P)$ and $b \in K(Q)$ and construct a line L(a,b) through a,b. Let θ be the direction of this line. Now construct VH(P,θ) and VH(Q,θ). Since $P \in$ VH(P,θ) and $Q \in$ VH(Q,θ), it is sufficient to show that the visibility hulls can be separated. First we note that VH(P,θ) and VH(Q,θ) are *star-shaped* with respect to a and b, respectively, by lemma 4.2. Next we show that the interiors of VH(P,θ) and VH(Q,θ) do not intersect. For, let $x \in$ int(Q). Then clearly x cannot lie in P. Furthermore, if x lies in a pocket of P then by lemma 3 x is not visible from L(a,b) $\cap$ ext{VH(P,θ)] and thus not visible from b which contradicts the star-shapedness of Q. On the other hand if x lies in a pocket of Q it cannot lie in P as it would contradict the star-shapedness of P with respect to a. Neither can it also lie in a pocket of P as it would contradict the star-shapedness of VH(Q,θ) with respect to b. Similar arguments hold for the case where $x \in$ int(P). Thus, VH(Q,θ) and VH(P,θ) are two non-intersecting polygons monotonic in a common direction $\theta + \Pi$, and by lemma 4.1, the result follows. Q.E.D.

Theorem 4.3 also provides us with an algorithm for finding a direction of translation. All we need to do is to find a point $a \in K(P)$ and a point $b \in K(Q)$. This can be done with the linear-programming algorithm of Dyer [24].

Theorem 4.4: Given two *star-shaped* n-gons, a direction for separating them can be determined in O(n) time.

Actually, *all* directions of separability by *translation* determined by points in the kernels can be found in O(n) time. Since the kernels are convex polygons, all directions determined by two points $a \in K(P)$ and $b \in K(Q)$ lie in a *cone* defined by the *critical lines of support* between K(P) and K(Q). The kernels can be computed in O(n) time with the algorithm of Lee and Preparata [25], and the *critical lines of support* can be found in O(n) time using the “rotating calipers” [22]. It is obvious that the above set of directions for separability is not the complete set. *All* directions for separating two arbitrary simple polygons by a single translation can however be determined in O(n log n) time [26].

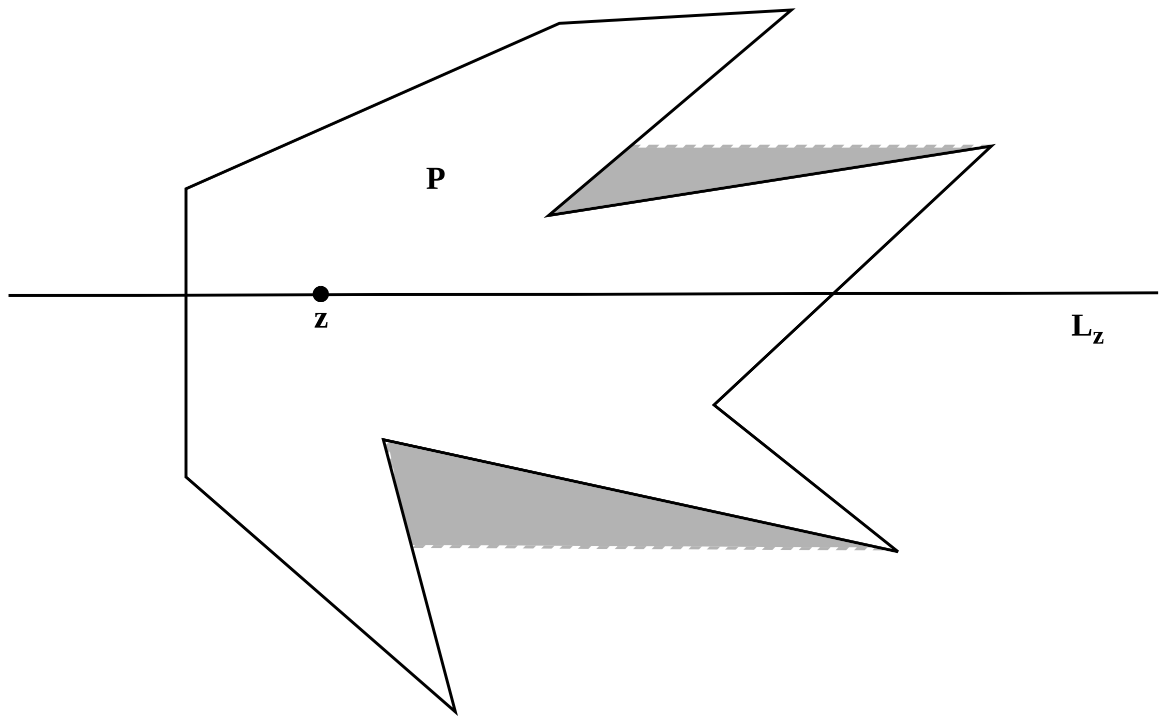


Figure 12: Illustrating the proof of lemma 3.

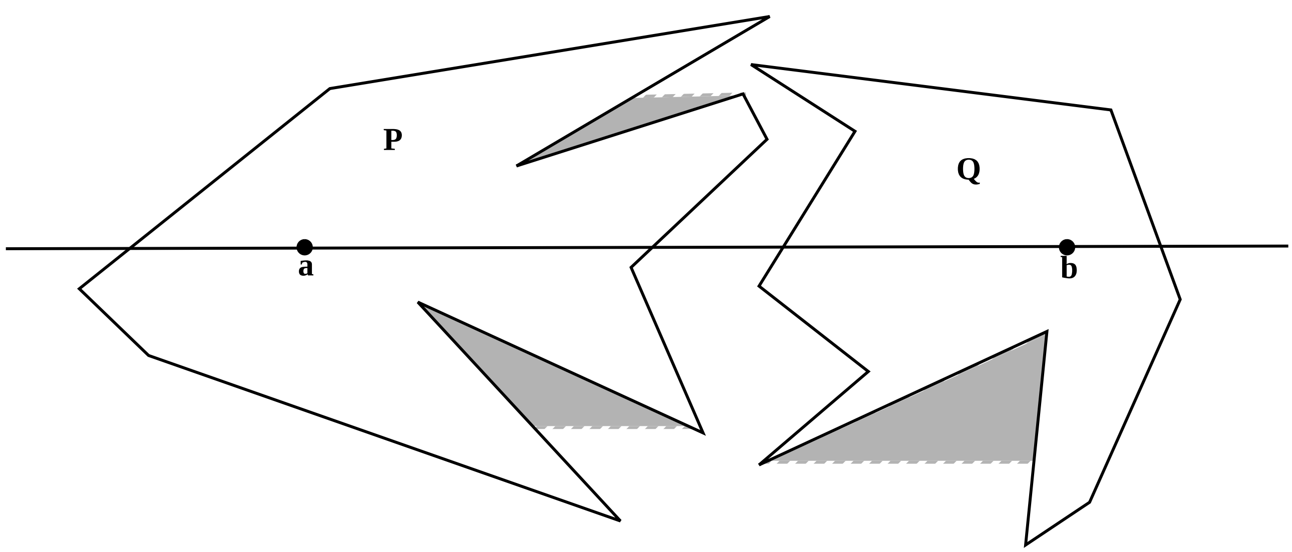


Figure 13: Two star-shaped polygons are separable with a single translation.

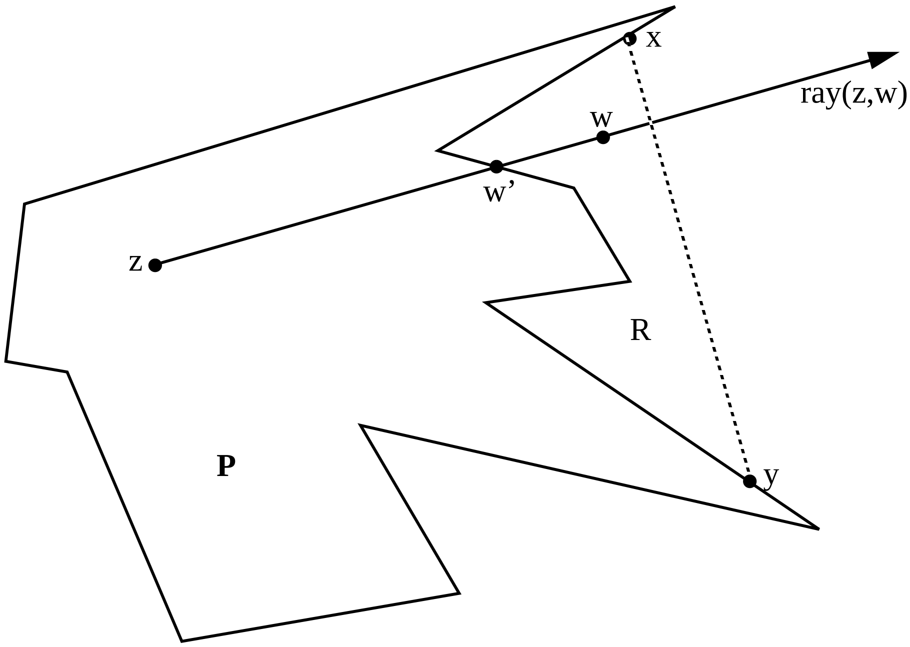


Figure 10: Illustrating the proof of lemma 2.

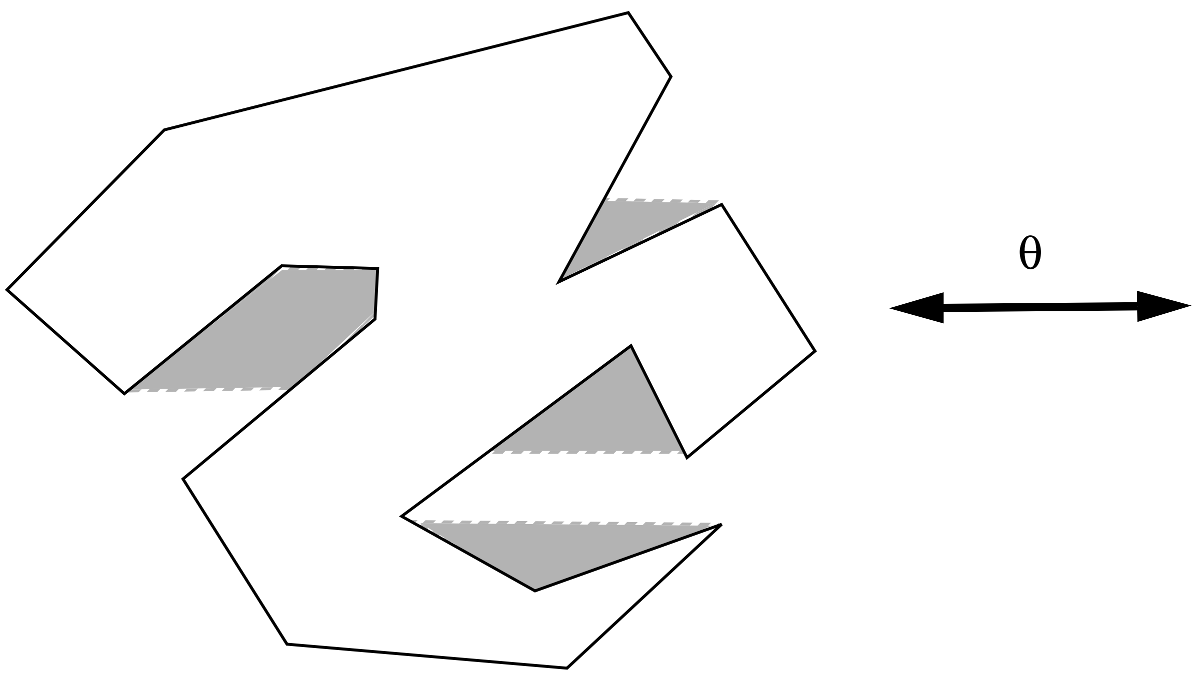


Figure 11: The visibility hull of P in direction theta.

Theorem 4.1: (G. Strang [21]) A convex polygon P can pass through a slit of length L, if and only if, $w \leq L$.

We remark here that the width of a convex n-gon can be computed in O(n) time using the "rotating calipers" [22], [23] and thus we obtain the following theorems.

Theorem 4.2: Given a convex n-gon P and a slit I of length L, whether or not P can pass through I can be determined in O(n) time.

We are now ready to show that two *completely externally visible* polygons can interlock. One such example is illustrated in Figure 9. First we construct a polygon P with a large deficiency polygon compared to its lid of length L. The lid plays the role of the slit in Figure 8. Next we construct a convex polygon of width $w > L$ in the interior of the deficiency polygon. From Theorem 4.1 it follows that P and Q are interlocked. Thus we see that in terms of movable separability *completely externally visible* polygons offer no additional freedom over arbitrary simple polygons in this sense.

4.3: Star-shaped polygons

We can ask the same question as in section 4.2 for star-shaped polygons. It turns out that *two star-shaped* polygons are always separable with a *single translation,* which we now prove.

The following lemma is proved in [12].

Lemma 4.1: Two polygons *monotonic* in a common direction θ, are movably separable with a single translation in a direction *orthogonal* to θ.

Lemma 4.2: If P is *star-shaped* with respect to a point z and if x,y are two points on bd(P) such that $(x,y) \in$ ext(P), then the polygon $P^* = P \cup R$, where R is the bounded exterior of P is also *star-shaped* with respect to z.

Proof: (Refer to Figure 10) Let w be a point in R and construct the half line emanating from z and passing through w, denoted by ray(z,w). Since z lies in the *kernel* K(P), ray(z,w) can intersect bd(P) only at one point, say w'. Now, the creation of P* involves substituting the boundary of P between x and y by [x,y]. That for all $w \in$ R, the corresponding w' must lie on this section of bd(P), follows from the fact that x and y are visible from z. Since w' is removed in the creation of P* it follows that w is visible from z in P*. Since this is true for all w, P* remains star-shaped with respect to z. Q.E.D.

Definition: Given a simple polygon P and a direction θ, the *visibility hull of P in direction* θ, denoted by VH(P,θ), is the set obtained by taking the union of P with all line segments [a,b] parallel to θ such that $a,b \in$ P. Note that VH(P,θ) is monotonic with respect to a direction orthogonal to θ. The visibility hull of P is the union of P with some "pockets" as illustrated in Figure 11, and can be interpreted as the region bounded by the portions of P visible from $\pm\infty$ in direction θ. Note that VH(P,θ) may have new vertices which are not vertices of P.

Definition: The *visibility deficiency* of P, denoted by VD(P,θ), in direction θ is the closure of the set-difference between P and the *visibility hull* of P.

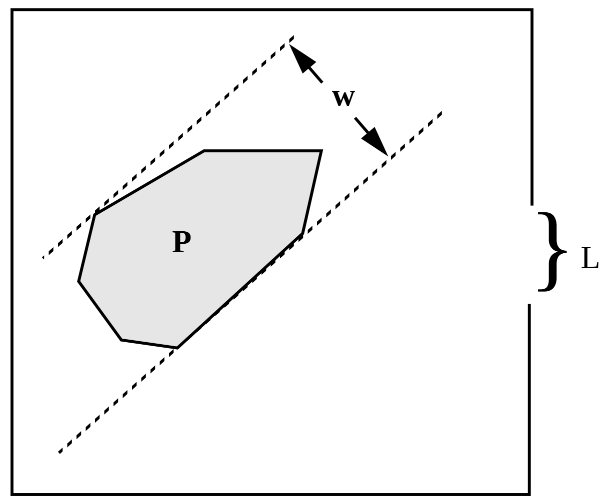


Figure 8: Passing a convex polygon through a slit.

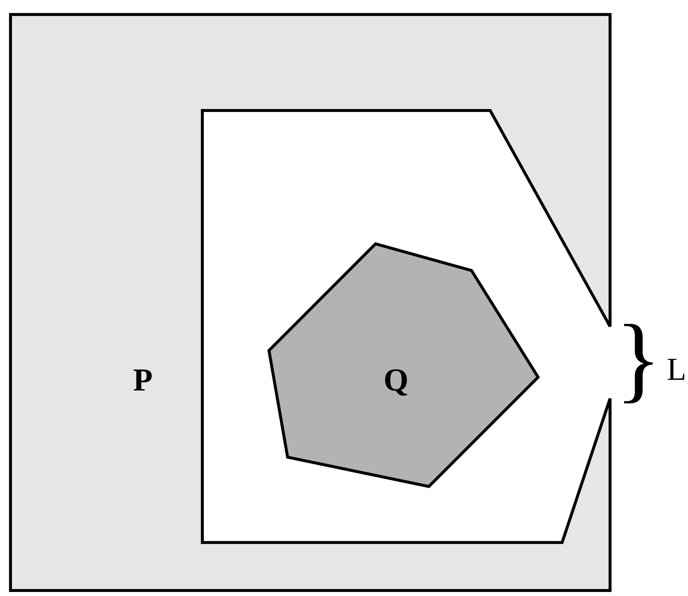


Figure 9: Two completely externally visible polygons can interlock.

Definition: A vertex p_i of a polygon P is said to be *unimodal* if the euclidean distance function $d(p_i, p_{i+1}), d(p_i, p_{i+2}),..., d(p_i, p_{i-1})$ has only one local maximum.

Definition: A polygon P in *unimodal* if every vertex of P is unimodal.

Note that at first glance there appears to be a close relationship between *unimodal* and *convex* polygons. In fact there is no relationship whatsoever. For a detailed treatment of this topic and other definitions of unimodality the reader is referred to [18].

Definition: A polygon P is *monotone* if there exists a direction θ such that the two opposite extreme vertices in direction θ partition the polygon into two polygonal chains each of which, when traversed, yields a monotonically increasing projection onto a line in direction θ.

4. Some Movable Separability Problems in the Plane

4.1: Isothetic rectangles

Consider following the uppermost path in the graph of Figure 7 starting with *isothetic rectangles*. As mentioned earlier, Guibas and Yao [8] have show that for a set of n isothetic rectangles all directions θ admit a *translation ordering* and such an ordering can be computed in O(n log n) time. This property was shown by Guibas and Yao [8] to remain true for sets of convex polygons. Thus there appears to be no basic difference in the separability properties between these two families of polygons. Actually, there is a slight difference worth mentioning.

Referring to Figure 2 we note that sorting the projections of the *support vertices* on *l* to obtain a translation ordering in direction *l* can be viewed as *sweeping* a line in the opposite direction and identifying the support vertices in the order in which the “sweep line” traverses them. This idea will be referred to as the *line-sweep heuristic* and it is, in fact, the first idea that a person invariably proposes for obtaining a translation ordering of convex polygons. Although the *line-sweep* heuristic can fail even for rectangles it is interesting to ask whether there exist classes of objects for which it is guaranteed to produce a valid ordering, In fact, one such class is precisely the set of *isothetic rectangles* when *l* is restricted to the ±x and ±y directions. In this particular case the first rectangle traversed by the “sweep line” can always be moved first. It follows that for these four directions a translation ordering can be easily obtained by *sorting* the corresponding edges of the rectangles. Thus, although the order of the complexity is not changed, the algorithm (pure and simple sorting) is much simpler than the O(n log n) algorithms of either Guibas and Yao [8] or Ottmann and Widmayer [9]. Other examples where the “sweep line” technique works successfully are given in [20].

4.2: Completely external visible polygons

Referring to Figure 7 and moving along the graph from *convex* to *completely external visible* polygons we encounter a dramatic change in movability properties. First we consider the problem of passing a convex polygon through a slit (see Figure 8). Let $P = (p_1, p_2,..., p_n)$ be a convex polygon. Let L be the length of the slit.

Definition: The *width* w of a convex polygon P is the *minimum* distance between parallel lines of support of P.

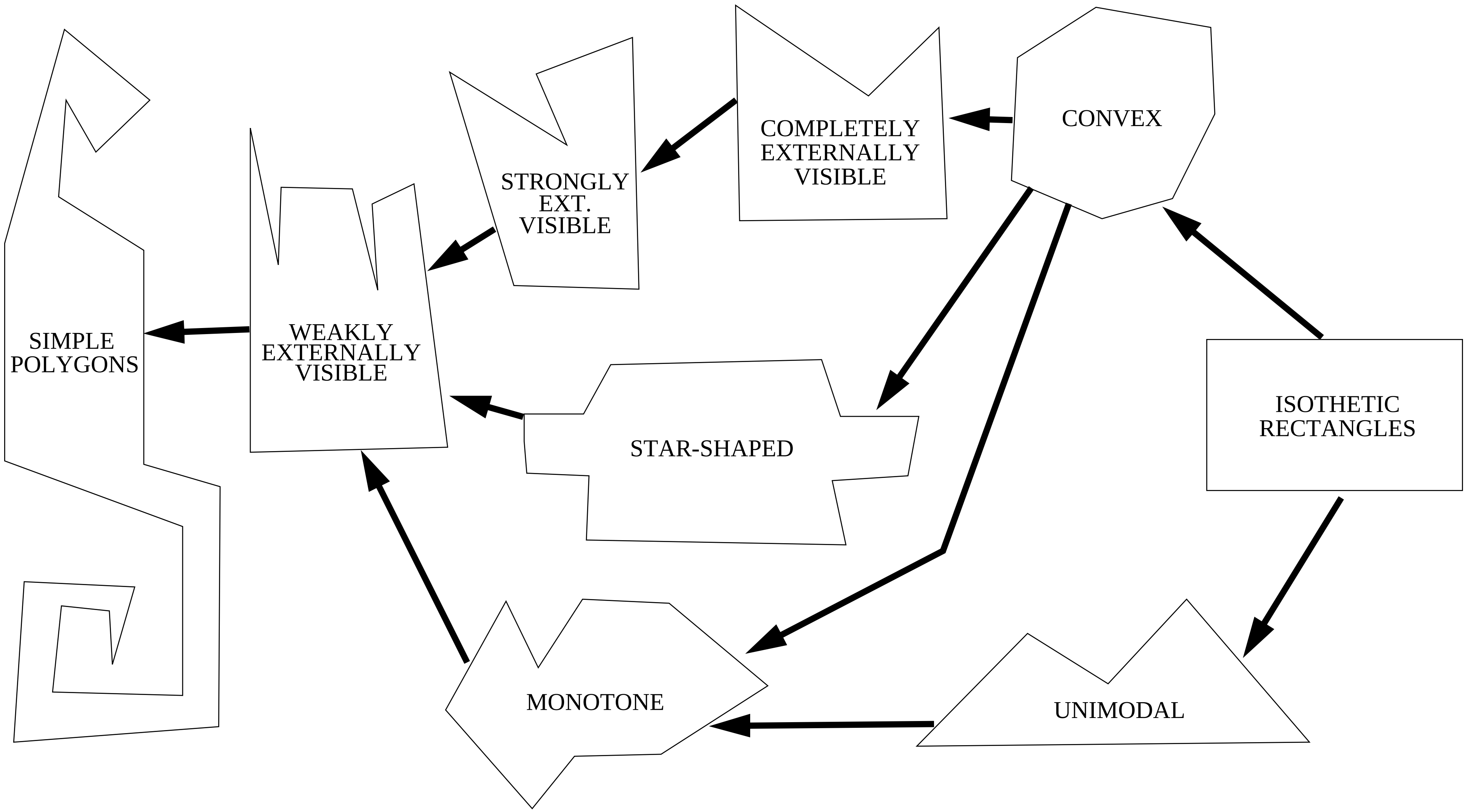


Figure 7: A hierarchy of simple polygons.

3. A Hierarchy of Simple Polygons

Let P = (p_1, p_2,..., p_n) be a simple polygon. i.e., we are given a list of its vertices, in clockwise order, along with their cartesian coordinates. We assume the polygon is in *standard form*, i.e., the vertices are distinct and no three consecutive vertices are collinear. A pair of vertices, say p_i p_{i+1}, defines the i^{th} edge of P. The sequence of vertices and edges forming the boundary of a polygon P, and denoted by bd(P), partitions the plane into two open regions: one bounded, termed the *interior* of P and denoted by int(P), and the other the unbounded *exterior* of P, denoted ext(P). When we consider a collection of polygons, each polygon is assumed to contain n vertices to simplify notation.

We saw with the example of Figure 2 that any finite collection of *rectangles*, no matter how large, is movably separable under translations. In fact, all fixed directions admit a translation ordering. On the other hand, from Figure 3 (c) we see that two arbitrary simple polygons are sufficient to form an *interlocked* set. Now, *rectangles* and *simple polygons* are extremes among a hierarchy of families of simple polygons of varying degrees of shape complexity. An example of hierarchy of nine families of polygons is given by the directed graph in Figure 7. A node in this graph represents a family of polygons. A directed path connects node A to node B (A $\longrightarrow$ B) if, and only if, polygons belonging to family A also belong to family B. For example, convex polygons are *star-shaped* but not vice-versa. It is thus interesting to follow different paths along this graph and to determine the places along the path where *movability properties* change. Actually, the graph in Figure 7 can be enlarged by the inclusion of a score of additional families of polygons [17] but this set will suffice to illustrate the point. Before considering movability we provide some definitions.

Definition: The *convex deficiency* of a polygon P is a set of *deficiency polygons* D_1, D_2,..., D_k obtained by subtracting int(P) from the convex hull of P and deleting the edges of P that are also edges of the convex hull of P. (Deficiency polygons are also more affectionately termed *pockets*.)

Definition: The edge of a pocket of P which is not also an edge of P is the *lid* of the pocket.

Definition: A polygon P is *completely visible* from edge e if for every point x in e and every point y in P, the line segment [x,y] lies in P.

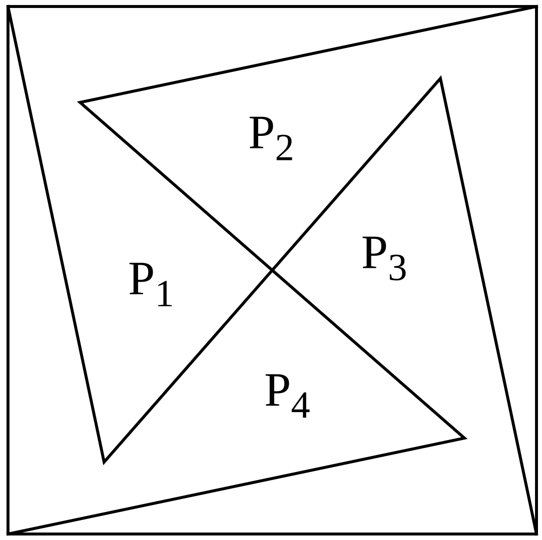
Definition: A polygon P is *completely externally visible* if every deficiency polygon of P is completely visible from its *lid*

Definition: A polygon P is *strongly visible* from edge e if there edge-exists a point x in e such that for all y in P, the line segment [x,y] lies in P.

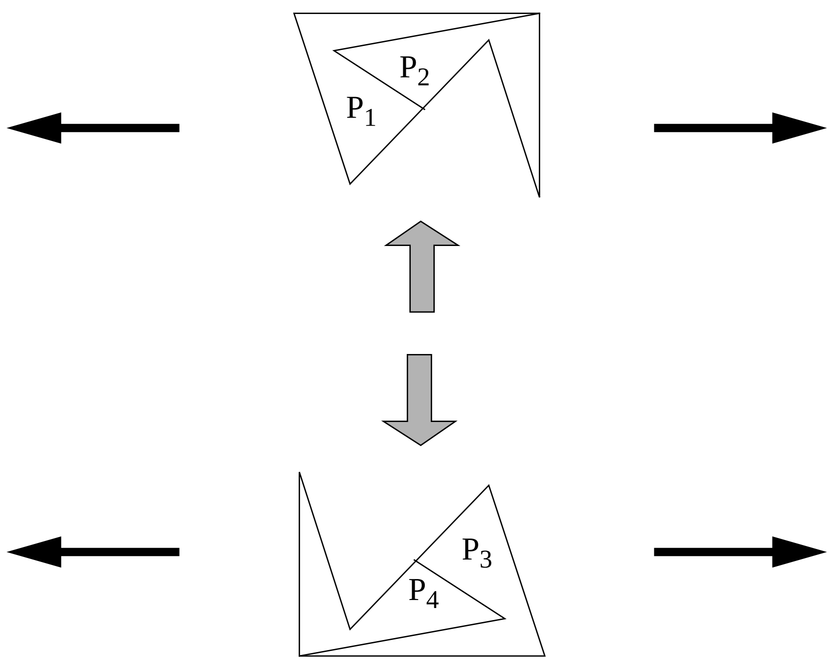
Definition: A polygon P is *strongly externally visible* if every deficiency polygon P is strongly visible from its lid.

Definition: A polygon P is *weakly externally visible* if every deficiency polygon is *edge-visible* from its lid

Definition: A polygon P is said to be *star-shaped* if there exists a region K in P, termed the *kernel* of P, such that for all x$\in$ K and all y$\in$ P the line segment [x,y] lies in P.



(a) Initial configuration.



(b) After simultaneous translation of P_1 and P_2 in the +y direction.

Figure 6: A set of simultaneously separable polygons.

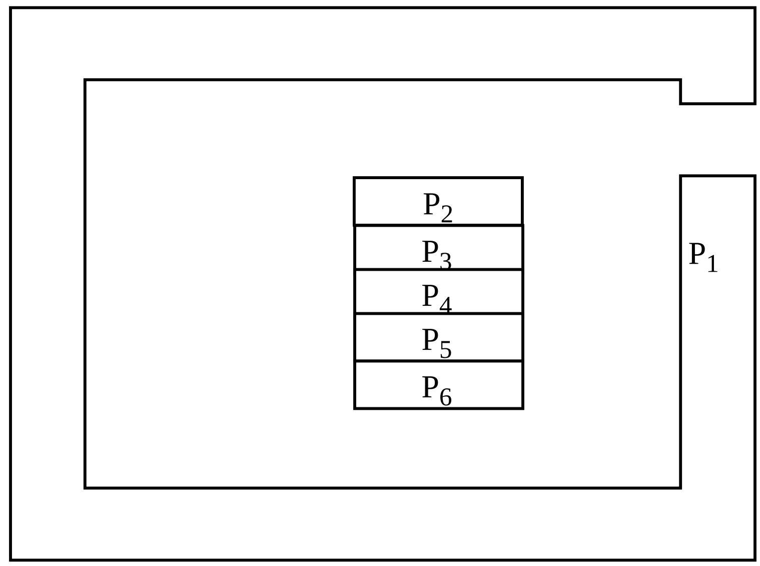


Figure 4: A set of sequentially movably separable polygons.

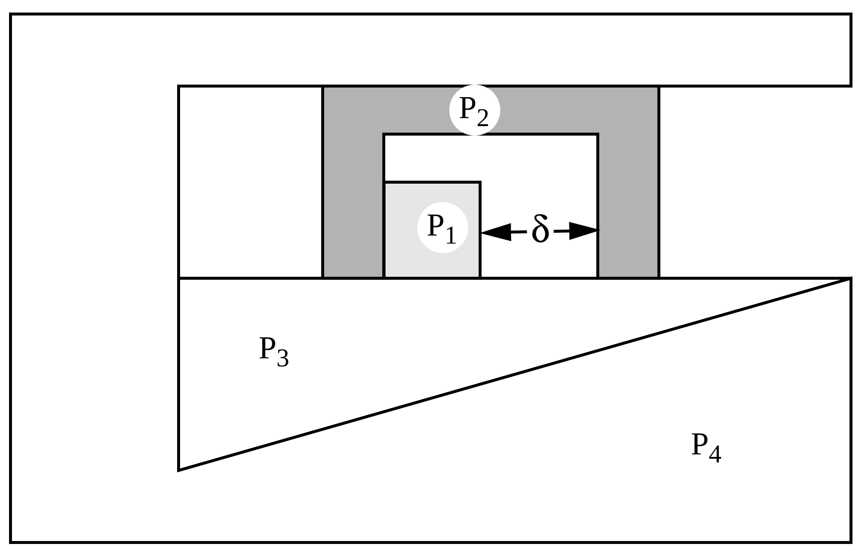


Figure 5: A set of four movably separable edge-visible polygons which is not sequentially separable.

translate P_1 and P_2 as a block in the +y direction until the convex hull of P_1 and P_2 does not intersect the convex hull of P_3 and P_4. Subsequently P_1 and P_4 can be translated in the -x direction and P_2 and P_3 can be translated in the +x direction. This example leads us to the following definition.

Definition: A set of objects IP is *simultaneously movably separable* if **P** is separable *only* by moving a subset of **P**, of cardinality greater than one, *simultaneously*. (Note that a motion with velocity zero clearly cannot count as a motion).

Definition: A set of objects **P** which is not movable separable is said to be *interlocked*.

These concepts are illustrated in Figure 3. In Figure 3 (a) we see that linear separability implies movable separability. For example, either P or Q can be translated to infinity in a direction parallel to the line of separability. In Figure 3 (b) we note first, that movable separability does not imply linear separability. This follows from a well-known theorem in convexity theory [16] which stated that "two sets are *linearly separable* if, and only if, their convex hulls do not intersect and the fact that convex hull intersection need not hinder movable separability. Secondly, Figure 3 (b) also illustrates the obvious fact that movable separability implies nonlinear separability. Finally, Figure 3 (c) shows that nonlinear separability obviously need not imply movable separability. In fact even a low-order polynomial discriminant function [15] such as a *quadratic* does not imply movable separability. In summary we have:

linear separability ===> movable separability ===> nonlinear separability.

If we are dealing with a collection of more than two objects then a variety of different types of movable separability is possible. Let $\mathbf{P} = \{P_1, P_2,..., P_n\}$ be a collection of objects. Denote an ordering of $\mathbf{P}$ by $\mathbf{P}' = \{P_{(1)}, P_{(2)},..., P_{(n)}\}$, and let $\mathbf{P}'_{(i)} = \{P_{(i)}, P_{(i+1)},..., P_{(n)}\}$.

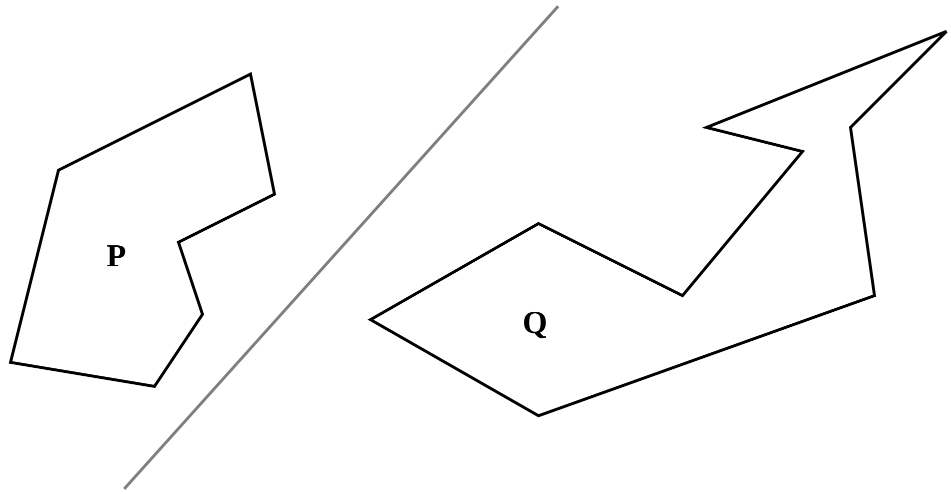
Definition: A set of objects $\mathbf{P}$ is *sequentially movably separable* if there exists an ordering $\mathbf{P}'$ such that for i = 1,2,...,n-1, $P_{(i)}$ can be moved an arbitrary distance away from $\mathbf{P}'_{(i+1)}$ without colliding with any object in $\mathbf{P}'_{(i+1)}$.

Figure 4 illustrates a set of *sequentially movably separable* polygons. Here $\mathbf{P}' = \{P_2, P_3, P_4, P_5, P_6, P_1\}$ and P_2 through P_6 can be moved to infinity with two translations each.

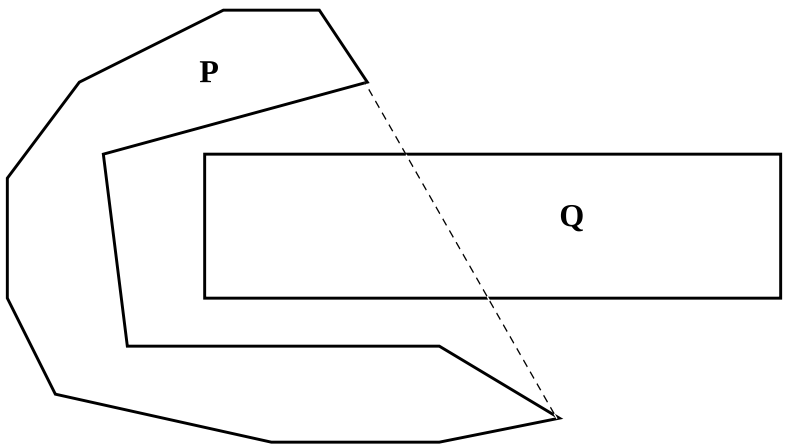
Definition: If there exists a set of motions on a collection of objects $\mathbf{P}$ such that for each object P_i, i = 1,2,...,n, P_i can attain an arbitrary large distance from $\{\mathbf{P} - P_i\}$ without collisions, then $\mathbf{P}$ is said to be *movably separable*.

Figure 5, a variation of an example due to Chazelle, et al., [13], illustrates a collection of four *edge-visible* polygons which are *not sequentially movably separable* and yet they are *movably separable*. Note that in Figure 5 no object can be moved to infinity without colliding with the remaining objects. Yet if we allow repeated alternating translations of P_1 and P_2 in the +x direction then eventually P_1 and P_2 lie outside the convex hull of P_4 and all four polygons can be translated to infinity. Note however that by making the separation δ, between P_1 and P_2, arbitrarily small we can require an arbitrarily large number of motions for achieving separation, independent of the number of objects. Chazelle, et al. [13] also give an example where the number of motions is *exponential* in the number of objects. (A polygon P is *edge-visible* if it contains an edge e such that for every point x in P there exists a point y in e such that the line segment [x,y] lies in P. Here [x,y] denotes *closed* line segment. Similarly, an *open* line segment will be denoted by (x,y)).

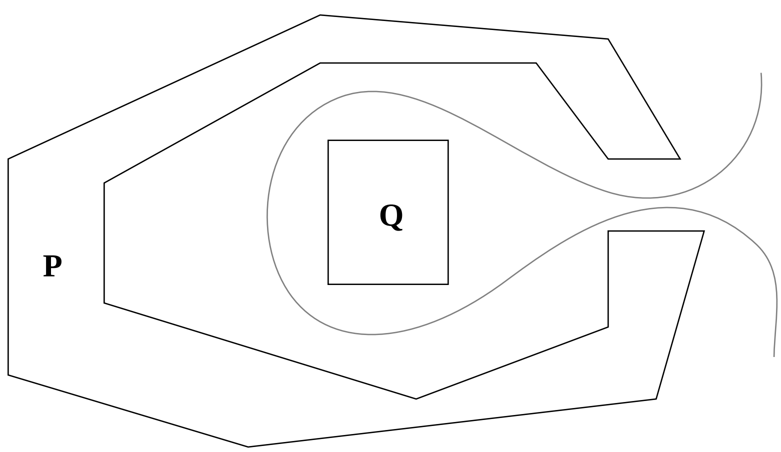
Note that in the example if Figure 5 an object is moved as often as required. However, separability is still attained by moving only *one* object at a time. It is possible that a set of objects is *separable* but *only if simultaneous* motions of several objects are required. One such example is illustrated in Figure 6. This set is not *sequentially movably separable*. Neither is it separable with repeated individual motions, and yet it is separable under simultaneous motions. In fact there exists an infinite number of non-trivial sets of motions that will separate this set. One example is to first



(a) Linearly separable polygons.



(b) Polygons not linearly separable but movably separable.



(c) Polygons not movably separable but nonlinearly separable.

Figure 3: Illustrating the relation between linear separability, movable separability and nonlinear separability.

fact they prove this also for a set of n convex polygons. Thus we will say that convex polygons in the plane exhibit the *translation-ordering property*. A simpler O(n log n) algorithm for solving this problem was later discovered by Ottman and Widmayer [9].

This problem was generalized to consider other types of polygons, and motion besides simple translation in [10] - [12]. One class of problems that results when convexity is relaxed concerns “interlocking polygons”. Thus one class of problems considered in [10] - [12] deals with determination of whether a given collection of polygons is “movably separable” in a specified sense. Some of the results of [10] - [12] were also independently obtained by Chazelle et al. [13] for the special case of *isothetic* polygons.

2. Movable Separability of Sets: Definitions

In this section we define some notions of *movable* separability and relate them to the well-known concepts of *linear* and *nonlinear* separability of sets [14] - [15], [19].

Definition: An object P can be *moved* from position A to position B if there exists a finite sequence of translations and rotations (possibly simultaneous) of P that carries P from A to B.

Definition: Let two objects P and Q be moved during a time interval starting at t_1 and ending at t_2. We say that a *collision* occurs between P and Q if there exists a time t such that $t_1 \leq t \leq t_2$ and the *interiors* of P and Q intersect.

Definition: Two objects are *movably separable* if one of them can be *moved* an arbitrary distance *without colliding* with the other.

If two objects are *not* movably separable then they are said to be *interlocked*. Note that a variety of definitions of movable separability are immediately apparent by specifying the type of motion considered. The simplest might be a *single translation*. A more complicated case may allow many translations but no rotations, and going further, any number of both types of displacements may be allowed but not simultaneously.

Note that when we speak of *distance* between two objects or between one object and a collection of objects we are measuring the distance between two sets, say S_1 and S_2. The distance used here refers to the minimum euclidean distance between an element in S_1 and an element in S_2.

It is interesting to see what the relation is between *movable separability* and the more well-known concepts of separability in discriminant analysis [14], [15], [19].

Definition: Let P and Q be two sets in Euclidean d-space, $\Re^d$. We say that P and Q are l*inearly separable* if there exists a hyperplane H that partitions $\Re^d$ into two half-spaces H(P) and H(Q) such that P is contained in H(P) and Q is contained in H(Q).

Definition: P and Q are *nonlinearly separable* if there exists a partition of $\Re^d$ into two non-intersecting regions R(P) and R(Q) containing P and Q, respectively.

1. Introduction

Consider a large one-room apartment with a corridor connecting it to the outside world, as illustrated in Figure 1. The corridor is assumed to have unit width and a right-angled corner. We can ask what is the figure of the largest area that can be moved out of the apartment and into the street through the connecting corridor. This is known as the *sofa problem* and has received some attention in the mathematics and computing literature [1] - [5]. Note that the room is as large as desired and we assume it is no obstacle to position the desired object into the first "wing" of the corridor. Thus we are really concerned only with getting by the corner of the corridor. An obvious lower bound on the solution to this problem is *unity* since a square of unit area can be moved out with two translations. However, we can find non-convex figures with areas as large as $(\pi/2) + (2/\pi) \cong 2.2074$ that can be moved out with a sequence of translations and rotations [5]. A variety of such problems present themselves if we vary the shape of the hallway (such as allowing left-angled as well as right-angled corners) and restrict the class of objects considered [5]. For example, we may ask for the largest (in some sense) *convex*, or *star-shaped* figure. We can also ask whether a given object or figure can be moved out of a specified apartment. Furthermore, we can ask for a sequence of motions that will free the object if such an action is possible. All these problems belong to a large family of problems which in this paper are termed *movable separability of sets*. While movable separability can be investigated for quite general sets, in this paper we consider the sets to be simple objects such as *line segments*, *circles*, and *simple polygons* in the plane or *spheres* and *polyhedra* in three dimensions. We also distinguish between *movable separability* problems and *collision avoidance* problems in *robotics* such as the *findpath* problem [6]. For a survey of those problems and their relation to computational geometry the reader is referred to [7]. It is difficult to formally define the class of problems labeled "movable separability". Nevertheless the problems considered here are in some sense more concerned with the notion of *separability* than the typical collision avoidance problem found in robotics.

As another example of a class of *movable separability* problems let us look at the problem of *translating* rectangles. Consider a set of non-intersecting rectangles in the plane whose sides are parallel to the x and y axes, as illustrated in Figure 2. Such rectangles are termed *isothetic*. A problem that arises in graphics and VLSI [8] is that of translating the entire collection by some common vector to a new location while respecting two constraints: first, the rectangles can only be moved one at a time and, second, during the entire process no *collisions* are allowed between the rectangles. A *collision* occurs if at some instant in time the interiors of the rectangles intersect. One problem that arises immediately is whether such a translation ordering property holds for all sets of rectangles and all directions. Another problem is the efficient computation of such an ordering if it exists. This is clearly a separability problem since each rectangle being translated is also being separated.

Referring to Figure 2, let *l* denote the desired direction of translation and construct that line of support to each rectangle perpendicular to *l* that maximizes the displacement in direction *l*. These lines of support intersect the rectangles A,B,C, and D at the *support vertices* a, b, c, and d, respectively. It is tempting, at first glance, to claim that a translation ordering can be obtained by sorting the projections of the *support vertices* on *l* and that therefore B can be moved first. Note that this is not the case, however, since this method would require D to move second which is impossible since it is blocked by C. Guibas and Yao [8] have shown that given a set of n rectangles and a direction *l*, a translation ordering always exists and can be computed in O(n log n) time. In

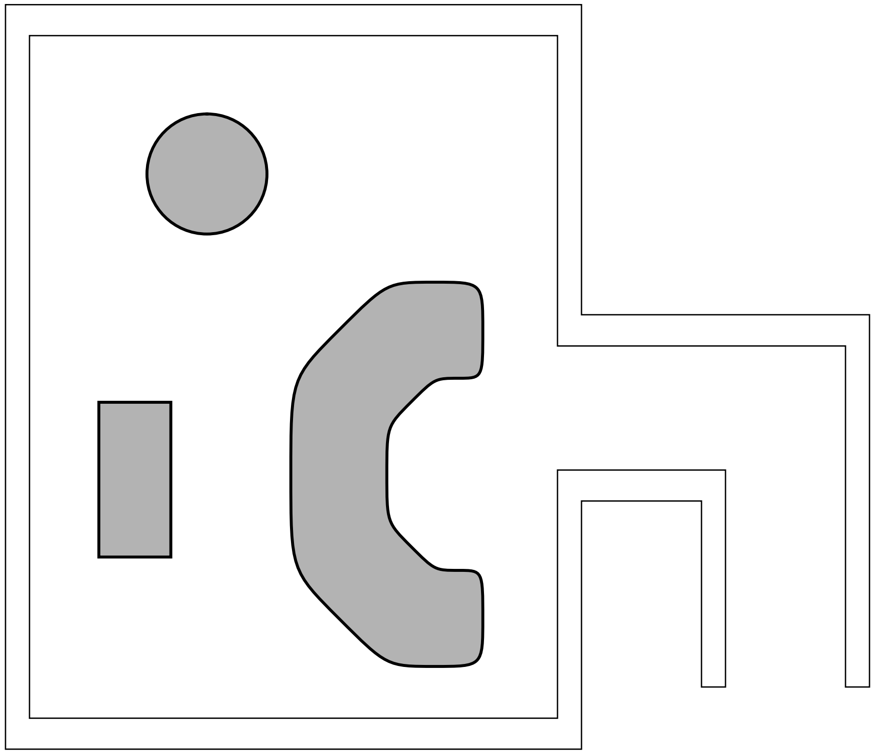


Figure 1: Moving Furniture out of the apartment.

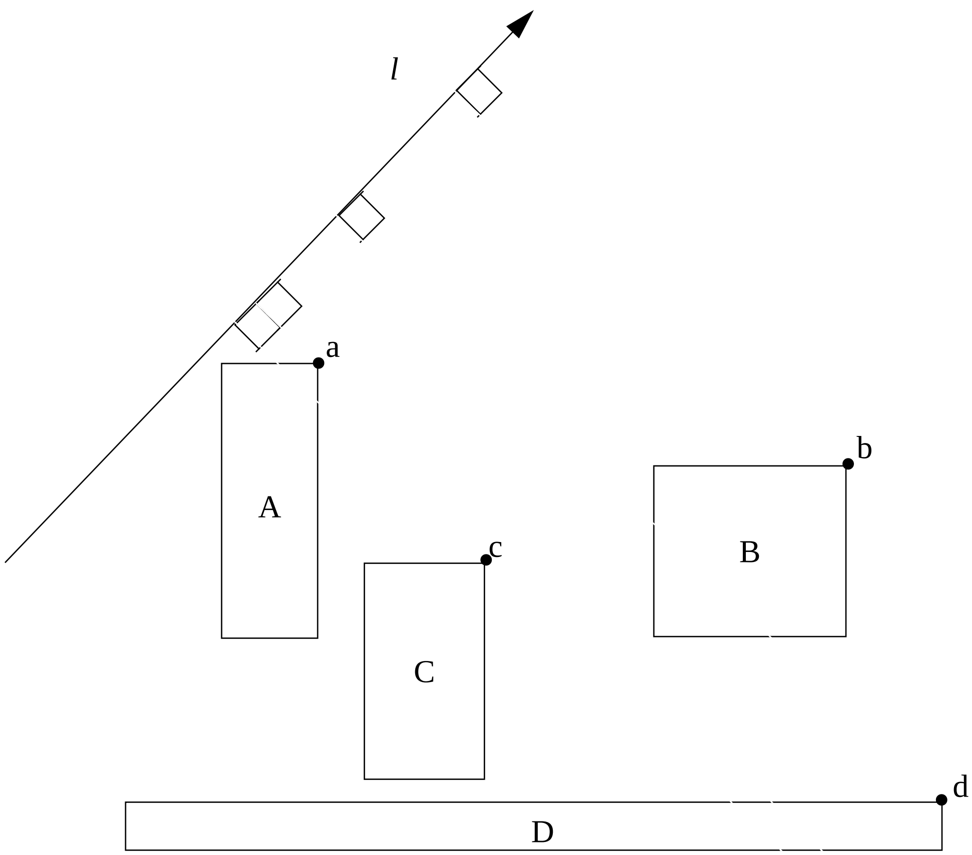


Figure 2: Rectangle D cannot be translated in direction *l* before rectangle C, and yet d occurs before c on line *l*.

Movable Separability of Sets

Godfried Toussaint
School of Computer Science
McGill University
Montreal, Quebec, Canada

Abstract

Spurred by developments in spatial planning in robotics, computer graphics, and VLSI layout, considerable attention has been devoted recently to the problem of moving sets of objects, such as line segments and polygons in the plane to polyhedra in three dimensions, without allowing collisions between the objects. One class of such problems considers the *separability* of sets of objects under different kinds of motions and various definitions of separation. This paper surveys this new area of research in a tutorial fashion, present new results, and provides a list of open problems and suggestions for further research.

Key Words and Phrases:

sofa problem, polygons, polyhedra, movable separability, visibility hulls, hidden lines, hidden surfaces, algorithms, complexity, computational geometry, spatial planning, collision avoidance, robotics, artificial intelligence.

CR Categories: 3.36, 3.63, 5.25. 5.32. 5.5

* Research supported by NSERC Grant no. A9293 and FCAR Grant no.EQ1678.