Polygons are Anthropomorphic

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We are concerned with a very special type of polygon in the Euclidean plane E^2 referred to as a *simple* (also *Jordan*) polygon. For any integer $n \geq 3$, we define a *polygon* or *n-gon* in the Euclidean plane E^2 as the figure $P = [x_1,x_2,...,x_n]$ formed by n points $x_1,x_2,...,x_n$ in E^2 and n line segments $[x_i,x_{i+1}]$, i=1,2,...,n-1, and $[x_n,x_1]$. The points x_i are called the *vertices* of the *polygon* and the line segments are termed its *edges*.

Definition: A polygon P is called a *simple* polygon provided that no point of the plane belongs to more than two edges of P and the only points of the plane that belong to precisely two edges are the vertices of P. A simple polygon has a well defined interior and exterior. We will follow the convention of including the interior of a polygon when referring to P.

Definition: (Meisters [Me2]) A vertex x_i of P is said to be a *principal* vertex provided that no vertex of P lies in the interior of the triangle $[x_{i-1},x_i,x_{i+1}]$ or in the interior of the diagonal $[x_{i-1},x_{i+1}]$.

Definition: (Meisters [Me1]) A *principal* vertex x_i of a simple polygon P is called an *ear* if the diagonal $[x_{i-1},x_{i+1}]$ that bridges x_i lies entirely in P. We say that two ears x_i and x_j are *non-overlapping* if $int[x_{i-1},x_i,x_{i+1}] \cap int[x_{j-1},x_j,x_{j+1}] = \emptyset$.

The following *Two-Ears* Theorem was recently proved by Meisters [Me1].

Theorem 1: (the *Two-Ears* Theorem, Meisters [Me1]) Except for triangles every simple polygon P has at least two *non-overlapping ears*.

Meisters' proof by induction is both elegant and concise. However, given that a simple polygon can always be triangulated allows a one-sentence proof [O'R]. *Leaves* in the *dual-tree* of the triangulated polygon correspond to *ears* and every tree of two or more nodes must have at least two *leaves*.

This theorem is quite applicable in many situations. For example it establishes that there exists a step-wise procedure for "shrinking" a polygon P down to a triangle by at each step deleting a vertex, say x_i, and inserting $[x_{i-1},x_{i+1}]$ in the place of $[x_{i-1},x_i,x_{i+1}]$ while ensuring that at each step we retain a simple polygon P'. In actual fact of course we need only a "*one ear*" theorem to carry out such a procedure. The method is evident: locate an ear in P and "cut it off," then locate an ear in the remaining polygon of one less vertex and cut it off, and continue this process until the remaining polygon is a triangle. It is obvious that such a procedure could also be used as an algorithm for computing a triangulation of P. However care must be taken in converting this idea into an efficient algorithm. A straightforward approach of implementing this notion can result in a very slow algorithm. To determine if a vertex is or is not an ear may take O(n) steps and we may have to visit O(n) vertices to find and cut off an ear. Therefore using a "brute force" approach we may have to perform $O(n^2)$ steps to cut off an ear and $O(n^3)$ steps to completely triangulate P in this manner. On the other hand algorithms exist for triangulating simple polygons in time O(n log n) [GJPT] and O(n log log n) [TV]. Once a triangulation is obtained the dual-tree can be determined in O(n) time. Finally an O(n)-time tree-traversal can prune off one *leaf* from the dual-tree at each step resulting in the cutting off of one *ear* from P at each step. It remains one of the most outstanding problems in computational geometry to determine if an O(n) time algorithm exists for triangulating arbitrary simple polygons.

One question that arises is whether the "inverse" of the previous procedure is possible, i.e., does there always exist a step-wise procedure for "inflating" a simple polygon P until it is as "fat-as-possible" by deleting vertices from P one-at-a-time so that at each step we retain a simple polygon? We answer this question in the affirmative by proving that every non-convex polygon contains at least one *mouth*, but first we must define *mouth* and make more precise what we mean by as "fat-as-possible."

Definition: A *principal* vertex x_i of a simple polygon P is called a *mouth* if the diagonal $[x_{i-1},x_{i+1}]$ is an *external* diagonal, i.e., the interior of $[x_{i-1},x_{i+1}]$ lies in the exterior of P.

The convex hull of a simple polygon P will be denoted by CH(P). The boundary (*bd*) of CH(P) is a convex polygon. We now have a precise definition of "as-fat-as-possible," i.e., P is inflated until it becomes the convex hull of P.

Theorem 2: (the *One-Mouth* Theorem) Except for convex polygons every simple polygon P has at least one *mouth*.

Proof: Construct the convex hull CH(P). Since P is non-convex there must exist edges on *bd*(CH(P)) that are not edges of P. Each such edge forms the "lid" of a "pocket" of CH(P). (refer to Fig. 1) We shall prove that in fact every such pocket yields a *mouth*. Let K_{ij} denote the pocket of CH(P) determined by vertices x_i and x_j of P. Clearly $K_{ij}= [x_i,x_{i+1},...,x_j]\cup[x_j,x_i]$ forms itself a simple polygon. By the *Two-Ears* Theorem K_{ij} must have two ears and since they are non-overlapping they cannot both occur at x_i and x_j. Therefore at least one ear must occur at x_k for $i < k < j$. Obviously such an *ear* for K_{ij} is a *mouth* for P. Q. E. D.

While the above step-wise procedure for "inflating" a polygon P by "gobbling-up" mouths provides an algorithm for computing the *convex hull* of P this is not the best way to tackle this prob-

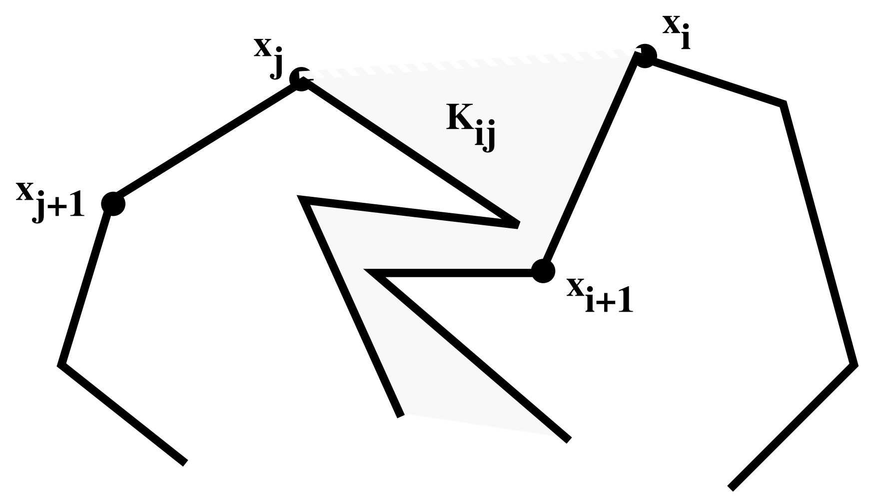

Fig. 1 Illustrating the proof of the *One-Mouth* Theorem.

lem. Triangulating P does not appear to help here and a straightforward approach to "gobbling-up" mouths leads to an $O(n^3)$ time algorithm. On the other hand several O(n) time algorithms for computing the convex hull of a simple polygon are known [MA], [GY], [To].

It is possible for a polygon to have many *ears* and only one *mouth* (Fig. 2 (a)) and also many *mouths* and only one *ear* (Fig.2 (b)). Note that care is needed when speaking of *mouths* and *ears* as well *exposed* vertices, i.e., vertices of P that are also vertices of CH(P). For example, Guggenheimer [Gu] states that a simple polygon has two *principal* vertices that are *exposed*. This is false and a counter-example due to Meisters [Me2] is illustrated in Fig. 2 (c). This figure also illustrates that polygons exist which have precisely *one mouth* and *two ears*. In fact, these notions suggest some interesting families of simple polygons. Recall that no O(n) time algorithm exists for triangulating an arbitrary simple polygon. However certain special classes of simple polygons such as *star-shaped* ones do admit O(n) time triangulation [To2]. We now define another such class of polygons.

Definition: A simple non-convex polygon P is called a *one-mouth* polygon provided it contains no more than one mouth.

Definition: A simple polygon P is called a *two-ear* polygon provided it contains no more than two ears.

Definition: A simple polygon P is called *anthropomorphic* provided it contains precisely two ears and one mouth. (see Fig. 2 (c))

These three classes of polygons exhibit a good deal of structure as exemplified by the following theorem.

Theorem 3: The dual-tree of every triangulation of a *two-ear* polygon is a chain.

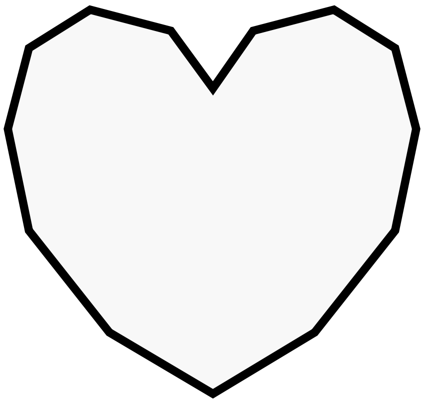

Fig. 2 (a) A polygon with only one *mouth* and many *ears*.

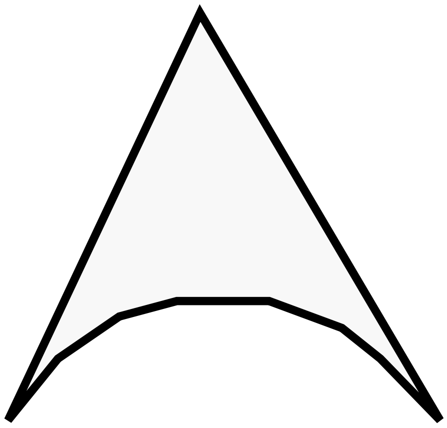

Fig. 2 (b) A polygon with only two *ears* and many *mouths*.

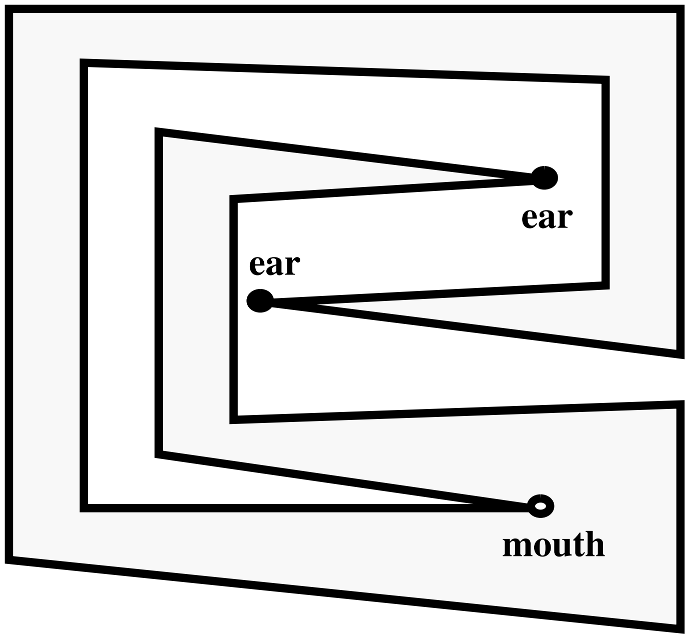

Fig. 2 (c) This polygon has precisely three *principal* vertices: two *ears* and one *mouth* and yet none of them are *exposed.*

Proof: Assume the polygon P has two ears and that the dual tree of some triangulation of P is not a chain. Then the tree must contain at least three leaves which is a contradiction. Q.E.D.

Theorem 3 allows us to triangulate the interior of a *two-ear* polygon of n vertices in O(n) time as follows. Consider any vertex x_i of P. It is an easy matter to find another vertex x_j such that $[x_j,x_i]$ is an internal diameter of P in O(n) time if indeed such a diagonal exists[Le]. Furthermore if such a diagonal does not exist then the diagonal $[x_{i-1},x_{i+1}]$ is guaranteed to exist[Le]. In either case this diagonal partitions the polygon P into two polygons P_1 and P_2 each of which can be triangulated in O(n) time starting at either $[x_j,x_i]$ or $[x_{i-1},x_{i+1}]$. It suffices to realize that each diagonal can be inserted with a constant number of local angle tests.

A similar procedure can be used to triangulate the exterior of a *one-mouth* polygon. First we can use an O(n) time algorithm for finding the convex hull of P [To1]. This will identify the two vertices x_i and x_j that form the "lid" of the pocket K_{ij} of CH(P). One of the two ears of K_{ij} must occur at either x_i or x_j and can then be identified in a constant number of steps (i.e., independent of n). Triangulation of K_{ij} can then proceed as in the case of the *two-ear* polygon.

We have therefore established the following theorems.

Theorem 4: A *one-mouth* polygon can be *externally* triangulated in O(n) time.

Theorem 5: A *two-ear* polygon can be *internally* triangulated in O(n) time.

Theorem 6: An *anthropomorphic* polygon can be *completely* triangulated in O(n) time.

One additional computational problem that is of interest here concerns the *recognition* of these types of polygons. For example, whether a simple polygon is star-shaped or not can be determined in O(n) time [LP]. By testing every vertex of a simple polygon to determine whether it is an ear or a mouth we can recognize *anthropomorphic* polygons in $O(n^2)$ time. However, using a more clever procedure we can reduce this complexity to O(n) [ST].

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