Computational Geometric Aspects of Musical Rhythm

Godfried Toussaint*
McGill University
Montréal, Québec, Canada

For our purpose a rhythm is represented as a cyclic binary string. Consider the following three 12/8 time ternary rhythms expressed in box-like notation: [x . x . x . x . x . x .], [x . x . x x . x . x . x] and [x . . . x x . . x x x .]. Here "x" denotes the striking of a percussion instrument, and "." denotes a silence. It is intuitively clear that the first rhythm is the most *even* (well spaced) of the three. Traditional rhythms have a tendency to exhibit such properties of evenness. Therefore mathematical measures of evenness find application in the new field of mathematical ethnomusicology [2], [17], where they may help to identify, if not explain, cultural preferences of rhythms in traditional music.

Clough and Duthett [3] introduced the notion of *maximally even sets* with respect to pitch scales represented on a circle. Block and Douthett [1] went further by constructing several mathematical measures of the amount of *evenness* contained in a scale. One of their measures simply adds all the interval arc-lengths (geodesics along the circle) determined by all pairs of pitches in the scale. However, this measure is too coarse to be useful for comparing rhythm timelines such as those studied in [13] and [15]. Using interval chord-lengths (as opposed to geodesic distances), proposed by Block and Douthet [1], yields a more discriminating measure, and is therefore a function that receives more attention. In fact, this problem had been investigated by Fejes Tóth [12] some forty years earlier without the restriction of placing the points on the circular lattice. He showed that the sum of the pairwise distances determined by n points on a circle is maximized when the points are the vertices of a regular n-gon.

One may also examine the *spectrum* of the frequencies with which all the durations are present in a rhythm. In music theory this spectrum is called the *interval vector* (or full-interval vector) [7]. For example, the interval vector for the clave *Son* pattern [x . . x . . x . . . x . x . . .] is given by [0,1,2,2,0,3,2,0].

Examination of such rhythm histograms leads to questions of interest in a variety of fields of enquiry: musicology, geometry, combinatorics, and number theory. For example, David Locke [9] has given musicological explanations for the characterization of the *Gahu* bell pattern, given by [x . . x . . x . . . x . . . x .], as "rhythmically potent", exhibiting a "tricky" quality, creating a "spiralling effect", causing "ambiguity of phrasing" leading to "aural illusions." Comparing the full-interval histogram of the *Gahu* pattern with the histograms of other popular 4/4 time traditional clave-bell rhythms leads to the observation that the *Gahu* is the only pattern that has a histogram with a maximum height of 2, and consisting of a single connected component of occupied histogram cells.

In 1989 Paul Erdős [5] asked whether one could find n points in the plane (no three on a line and no four on a circle) so that for every i, $i = 1, ...n-1$ there is a distance determined by these points that occurs exactly i times. Solutions have been found for $2 \leq n \leq 8$. A musical scale whose pitch intervals are determined by points drawn on a circle, and that has the property asked for by Erdős is known in music theory as a *deep* scale [7]. We will transfer this terminoly from the pitch domain to the time domain and refer to cyclic rhythms with the Erdős property as *deep* rhythms.

The analysis of cyclic rhythms suggests yet another variant of the question asked by Erdős. From the musicological point of view it is desirable (especially in African timelines) not to allow empty semicircles. Such constraints suggest the following problem. Is it possible to have k points on a circular lattice of n points so that for every i, $i = k_s, k_{s+1}, ..., k_f$ (s and f are pre-specified) there is a *geodesic* distance that occurs exactly i times, with the further restriction that there is no empty semicircle?

These problems are closely related to the general problem of reconstructing sets from interpoint distances: given a distance multiset, construct all point sets that realize the distance multiset. This problem has a long history in crystallography [8], and more recently in DNA sequencing [11]. Two noncongruent

*This research was partially supported by NSERC and FCAR. e-mail: godfried@cs.mcgill.ca

sets of points are called *homometric* if the multisets of their pairwise distances are the same [10].

The preceeding suggests that it would be desirable to be able to eficiently generate rhythms that contain prescribed histogram shapes, (such as *deep* rhythms) and to find approximations when such rhythms do not exist.

The problem of comparing two binary strings of the same length with the same number of one's suggests an extremely simple edit operation called a *swap*. A swap is an interchange of a one and a zero that are adjacent to each other in the binary string. The swap distance between two rhythms is the *minimum* number of swaps required to convert one rhythm to the other.

Consider two n-bit (cyclic) binary strings, A and B, represented on a circle (necklace instances). Let each sequence have the same number k of 1's. We are interested in computing the *necklace-swap-distance* between A and B, i.e., the minimum number of swaps needed to convert A to B, minimized over all rotations of A. This distance may be computed in $O(n^2)$ time by solving a linear time problem in each of the n rotated positions. The open problem is whether the $O(n^2)$ may be improved. In contrast, the *necklace-Hamming-distance* may be computed in $O(n \log n)$ time using the Fast Fourier Transform [6].

For additional discussion of the preceeding topics the reader is referred to [13], [15], [14], [17], [16], [4], and the references therein.

References

[1] Steven Block and Jack Douthett. Vector products and intervallic weighting. *Journal of Music Theory*, 38:21–41, 1994.

[2] M. Chemillier. Ethnomusicology, ethnomathematics. The logic underlying orally transmitted artistic practices. In G. Assayag, H. G. Feichtinger, and J. F. Rodrigues, editors, *Mathematics and Music*, pages 161–183. Springer, 2002.

[3] J. Clough and J. Douthett. Maximally even sets. *Journal of Music Theory*, 35:93–173, 1991.

[4] Miguel Díaz-Bañez, Giovanna Farigu, Francisco Gómez, David Rappaport, and Godfried T. Toussaint. El compás flamenco: a phylogenetic analysis. In *Proc. BRIDGES: Mathematical Connections in Art, Music and Science*, Southwestern College, Kansas, July 30 - August 1 2004.

[5] Paul Erdős. Distances with specified multiplicities. *American Math. Monthly*, 96:447, 1989.

[6] D. Gusfield. *Algorithms on Strings, Trees, and Sequences: Computer Science and Computational Biology.* Cambridge University Press, Cambridge, 1997.

[7] Timothy A. Johnson. *Foundations of Diatonic Theory: A Mathematically Based Approach to Music Fundamentals.* Key College Publishing, Emeryville, California, 2003.

[8] Paul Lemke, Steven S. Skiena, and Warren D. Smith. Reconstructing sets from interpoint distances. Tech. Rept. DIMACS-2002-37, 2002.

[9] David Locke. *Drum Gahu: An Introduction to African Rhythm.* White Cliffs Media, Gilsum, New Hampshire, 1998.

[10] Joseph Rosenblatt and Paul Seymour. The structure of homometric sets. *SIAM Journal of Algebraic and Discrete Methods*, 3:343–350, 1982.

[11] S. S. Skiena and G. Sundaram. A partial digest approach to restriction site mapping. *Bulletin of Mathematical Biology*, 56:275–294, 1994.

[12] L. Fejes Tóth. On the sum of distances determined by a pointset. *Acta. Math. Acad. Sci. Hungar.*, 7:397–401, 1956.

[13] Godfried T. Toussaint. A mathematical analysis of African, Brazilian, and Cuban *clave* rhythms. In *Proc. of BRIDGES: Mathematical Connections in Art, Music and Science*, pages 157–168, Towson University, MD, July 27-29 2002.

[14] Godfried T. Toussaint. Algorithmic, geometric, and combinatorial problems in computational music theory. In *Proceedings of X Encuentros de Geometria Computacional*, pages 101–107, University of Sevilla, Sevilla, Spain, June 16-17 2003.

[15] Godfried T. Toussaint. Classification and phylogenetic analysis of African ternary rhythm timelines. In *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, pages 25–36, Granada, Spain, July 23-27 2003.

[16] Godfried T. Toussaint. A comparison of rhythmic similarity measures. In *Proc. 5th International Conference on Music Information Retrieval*, pages 242–245, Barcelona, Spain, October 10-14 2004. Universitat Pompeu Fabra.

[17] Godfried T. Toussaint. A mathematical measure of preference in African rhythm. In *Abstracts of Papers Presented to the American Mathematical Society*, volume 25, page 248, Phoenix, January 7-10 2004. American Mathematical Society.