Simple Proofs of a Geometric Property of Four-Bar Linkages

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Abstract

We consider the relationship between the lengths of the two diagonals in the four-bar planar linkage. For convex and crossing linkages one diagonal increases if, and only if, the other decreases. On the other hand, for non-convex simple linkages one diagonal increases if, and only if, the other increases. We present simple elementary proofs of this geometric property. The proofs are simpler than existing published proofs, they illustrate the application of Decartes' principle of instantaneous centers of rotation, and they have pedagogical value.

1 Introduction

We consider the planar four-bar linkage or polygon of four sides where the angles of the vertices are allowed to change but the lengths of the edges are preserved during any motion. Such a mechanism is a fundamental component of many machines [5], [10], [13]. In addition it offers a relatively simple motion that forms a primitive for proving more general motions about complex chain linkages that serve as models for robot arms [12], knots [16], and molecules in polymer physics [23] and molecular biology [8]. In addition such primitive motions are used in proving geometric properties about polygons in general [18], [22], [12].

The planar four-bar linkage has been studied for a long time in the field of kinematics [5], where one of the bars (or its end points) was attached to the ground or other fixed parts of space. In this setting only three bars rotate. Indeed in the 19th century the four-bar linkage was called the *three-bar* linkage [17], [4]. Historically, the main aspect of linkages investigated was their capacity to generate a large variety of complicated curves from simple circular motion [7]. Depending on the relative lengths of the four bars, there are 27 different types of planar four-bar linkages that fall into 8 basic categories [13]. A curve is usually generated by the apex of a triangle (coupler) whose base is attached to the "floating" edge of the linkage. The Hrones-Nelson book [10] illustrates more than seven thousand different forms of the coupler curve! More recently there has been growing interest in describing and visualizing the configuration spaces of the linkages [11], [15].

The motion of the planar four-bar linkage is well understood and equations describing how one angle changes as a function of another are available [17], [4], [9]. Although the motions of the quadrilateral's edges *relative* to each other do not change when we fix different edges, the behavior with respect to the fixed edge changes radically. The process of changing the fixed edge in a given four-bar linkage is called *inversion* and gives much insight in the overall behavior of the linkage [5]. Additional insight can be obtained by analyzing the linkage with a type of inversion that does not appear to be used in kinematics but which is used frequently in robotics. Consider the four-bar linkage with vertices A, B, C and D in counterclockwise order as illustrated in Figure 1. Rather than fixing an edge of the linkage we fix a vertex, say A and a half line starting at A and proceeding through the vertex C opposite A. The motion is now determined by translating C along this half line. This motion is known as a *line-tracking* motion [12].

In the kinematics literature the linkage is analysed in terms of how one angle varies as a function of the angle of an adjacent vertex. For example, in Figure 1 (a) if we fix edge AD on the plane and rotate edge CD about D we are interested in what happens to the angle at A. However, one can equivalently analyse the linkage in terms of legths of diagonals rather than angles. Consider first a two-bar linkage chain ABC. Propositions 24 and 25 of Book 1 of Euclid's *Elements* state that the acute angle ABC increases if, and only if, the distance between its endpoints AC also increases [6]. This property of a two-bar chain linkage is also known as the *caliper lemma* [20]. Returning to Figure 1 (a) we see that angle D increases if, and only if, diagonal AC increases and angle A increases if, and only if, diagonal BD

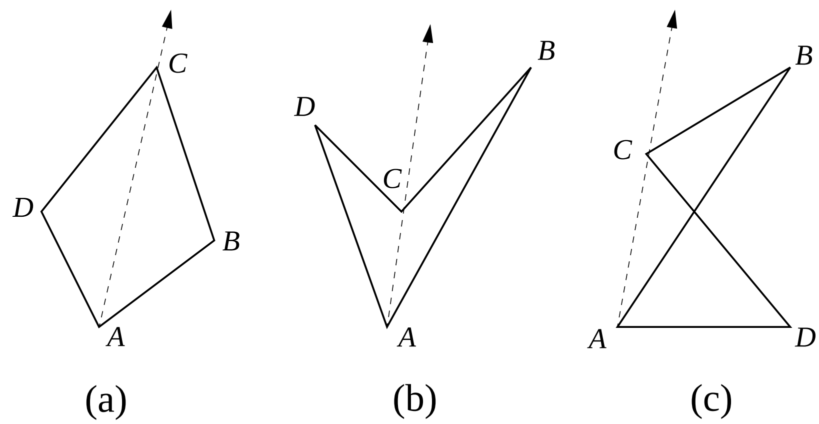

Figure 1: The three types of four-bar linkages. (a) convex, (b) concave and (c) crossing.

increases. Therefore instead of examining the behavior of angle A relative to angle D we may examine how diagonal DB varies as diagonal AC changes. In the following we will make copious use of this caliper lemma without bothering to refer to it every time it is invoked.

The fundamental geometric property of four-bar linkages may be expressed by the following theorem. We assume throughout that the linkage is not degenerate (is in general position) in the sense that no three of its vertices are collinear.

Theorem 1 *For* convex *and* crossing *four-bar linkages one diagonal increases if, and only if, the other decreases. In contrast, for simple* non-convex *linkages one diagonal increases if, and only if, the other increases.*

Although the statement of Theorem 1 may be novel, the result itself is not new. It is implicit in the complicated equations that relate adjacent angles of the linkage, and is contained in most kinematics books [13]. In fact, explicit elementary proofs of this theorem have been published for both the convex [1] and simple non-convex [3] cases. In this note we provide very simple proofs of the theorem for all three types of four-bar linkages.

2 The Convex Four-Bar Linkage

Aichholzer et al., [1] give an elementary proof of Theorem 1 for the case of convex linkages. Their proof however makes use of the Cauchy-Steinitz Lemma [19]. There are many published elementary proofs of this lemma but most are very long [21]. Even the shortest proof [19] selected for its elegance to adorn the pages of *Proofs from the BOOK* [2] adds unnecessary length, indirectness and complexity to the proof of Theorem 1. Here we provide two simpler shorter proofs that do not use the Cauchy-Steinitz Lemma.

2.1 First proof

In this proof we fix neither an edge nor a vertex. The proof works just as well for a "floating" linkage.

Consider a linkage $ABCD$ as in Figure 1 (a). Assume CA increases. Then the internal angles at B and D must increase. Now the internal angles at A and C cannot remain fixed, for then diagonal DB would remain fixed and the linkage would be rigid, contradicting the fact that AC increases. Therefore the internal angles at A and C either increase or decrease. Both angles cannot increase because then all four internal angles would increase violating the fact that the sum of the internal angles of any convex quadrilateral equals 2π. Finally, one angle cannot increase while the other decreases, for then by Euclid's caliper lemma, DB would both increase and decrease simultaneously, which is impossible. Therefore both angles must decrease causing DB to decrease.

2.2 Second proof

The second proof of Theorem 1 we present below is slightly longer than the first but it gives more insight into the motion. It also illustrates the approach we will use to prove the theorem for the more difficult cases of non-convex linkages. This approach uses Descartes's principle of instantaneous centers of rotation [14].

Let us consider vertex A to be fixed at the origin and assume without loss of generality that vertex D is located on the positive y-axis. Refer to Figure 2. We will prove the theorem by holding edge AD fixed in the plane. In this setting links AB and DC rotate about A and D, respectively. We want to prove that if AC increases then DB decreases. To this end we

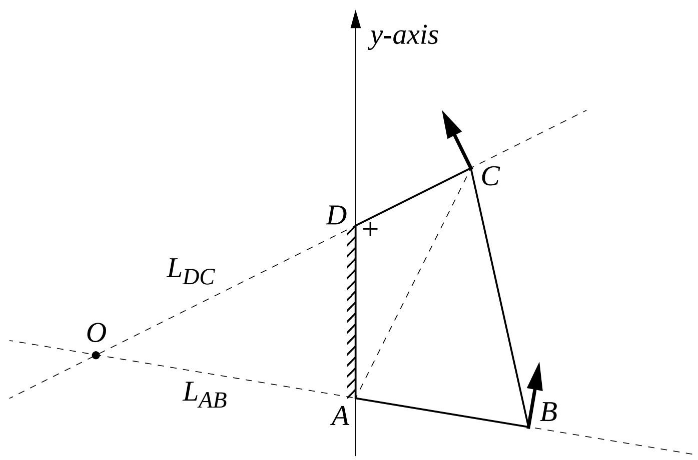

Figure 2: Illustrating the second proof with edge AD fixed by means of instantaneous centers of rotation.

examine the direction of instantaneous rotation of the "floating" edge BC. Since AC increases so does angle ADC. In the figures we indicate increasing and decreasing angles and diagonals by "+" and "−" signs, respectively. Therefore C rotates counter-clockwise with respect to D. The locus of centers of instantaneous rotation for C is the line L_{DC} that contains D and C. Similarly, since B rotates about A and moves orthogonally to AB the locus of centers of instantaneous rotation for B is the line L_{AB} that contains A and B. The intersection point O of these two lines is the instantaneous center of rotation for the edge BC. If O lies to the left of the y-axis as in Figure 2, then the edge BC rotates in a counter-clockwise manner with respect to O and so does B with respect to A. If O lies to the right of the y-axis the edge BC rotates in a clockwise manner with respect to O but B still rotates in a counter-clockwise manner with respect to A. If O lies at infinity, as happens when DC and AB are parallel, then CD translates in a direction orthogonal to L_{AB} with a $+y$ component. In all cases angle BAD decreases. Therefore diagonal DB decreases.

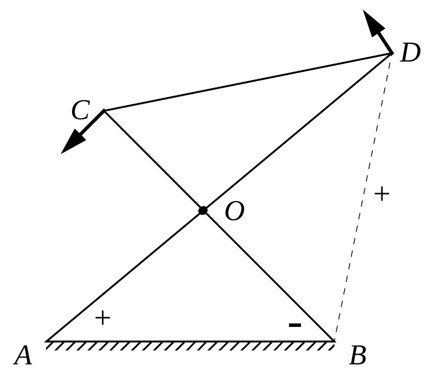

Figure 3: A crossing linkage with edge AB fixed and O as the instantaneous center of rotation of edge CD.

3 The Crossing Four-Bar Linkage

For the crossing four-bar linkage a short and simple proof is obtained by fixing one of the edges in the plane. Accordingly, let AB be the fixed edge and refer to Figure 3. We shall prove that if one of the diagonals, say BD, increases then the other diagonal AC must decrease. If BD increases then so does angle BAD. Since AB is fixed it follows that D rotates counter-clockwise with respect to A. Now the locus of centers of instantaneous rotations for D is the line containing AD, and for C the line containing CB. Therefore the instantaneous center of rotation for the "floating" edge DC is O, the intersection of the line segments AD and BC. Therefore edge CD rotates counter-clockwise with respect to O. Therefore C rotates counter-clockwise with respect to B. Since AB is fixed, angle ABC decreases and so does diagonal AC.

4 The Simple Non-Convex Four-Bar Linkage

An elementary but somewhat technical two-page proof of Theorem 1 for simple non-convex four-bar linkages was given by Biedl et al., [3]. In this section we first give a much simpler and shorter 8-line proof which, like theirs, holds one edge fixed in the plane. Then we prove the theorem for the more difficult line-tracking motion. Although the second proof is a longer

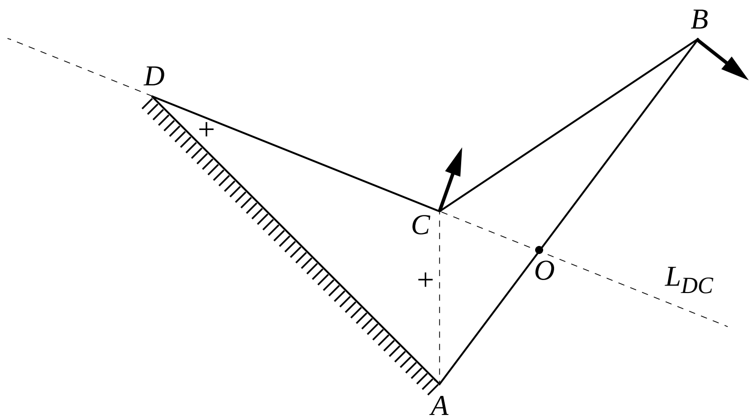

Figure 4: A simple non-convex linkage with edge AD fixed and O as the instantaneous center of rotation of edge BC.

case-analysis, it gives additional insight into the behavior of the linkage. In particular, it flushes out the more difficult case when all four edges rotate in the *same* direction with respect to the plane. Finally, we present a third more elegant proof which adds a third diad (two-bar linkage) forming a six-bar linkage and then applies the results of the convex and crossing linkages.

4.1 First proof - via edge-fixing motion

As before, let us fix edge AD in the plane and refer to Figure 4. We want to prove that if AC increases then so does DB. If AC increases then so does angle ADC. Since AD is fixed C rotates counter-clockwise about D. Since C is a concave vertex the line L_{DC} containing D and C (the locus of centers of instantaneous rotations for C) intersects the segment AB at O, the center of rotation for the "floating" edge BC. Therefore edge CB rotates clockwise with respect to O and B rotates clockwise with respect to A. Since AD is fixed angle DAB increases and so does diagonal DB.

4.2 Second proof - via line-tracking motion

Let us consider the simple non-convex four-bar linkage $ABCD$ in counter-clockwise order. As before, let C be the concave vertex, A be located at the

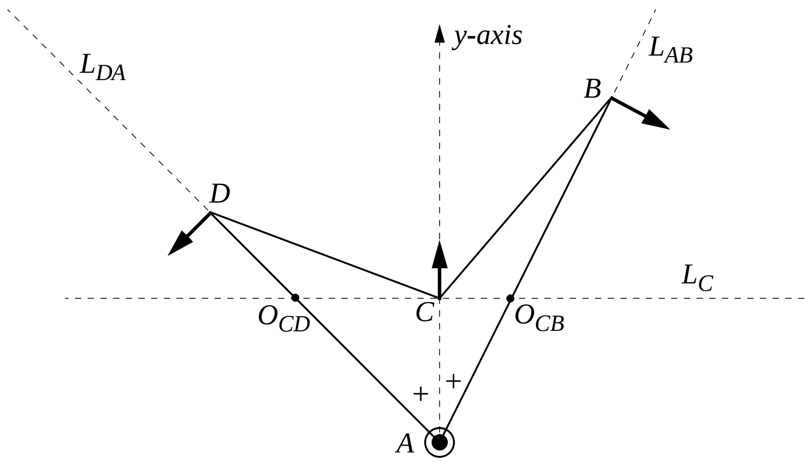

Figure 5: Illustrating the proof for the *line-tracking* motion for Case-1.

origin of the plane, and without loss of generality let C lie on the positive y-axis (see Figure 5). Since the linkage is not convex at least one of B or D must lie strictly above the line L_C which lies parallel to the x-axis and contains C. Without loss of generality assume B lies strictly above L_C.

A line-tracking motion moves C along the ray from A in the direction of C, i.e., the y-axis. We want to prove that if AC increases in this way with A fixed at the origin then the other diagonal BD must also increase. Note that the line L_C is the locus of instantaneous centers of rotation for C. We consider five cases for the location of D and analyze them in turn.

4.2.1 Case-1: D lies strictly above L_C

In this setting the bars AD and AB rotate about A and the loci of instantaneous rotations for D and B are the lines L_{DA} and L_{AB}, respectively. Since C translates upwards the edge CD rotates counter-clockwise with respect to O_{CD}, its instantaneous center of rotation. Therefore D rotates counter-clockwise with respect to A and angle CAD increases. Similarly, edge CB rotates clockwise with respect to O_{CB}, its instantaneous center of rotation. Therefore B rotates clockwise with respect to A and angle CAB also increases. Therefore angle BAD and diagonal BD also increase. In this case two edges rotate clockwise and two counter-clockwise.

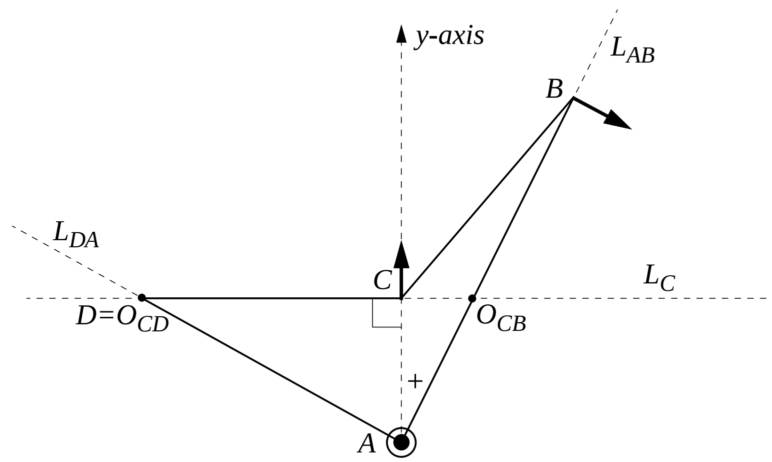

Figure 6: Illustrating the proof for the *line-tracking* motion for Case-2.

4.2.2 Case-2: D lies on L_C

Figure 6 illustrates case-2 when D lies on L_C. In this case the instantaneous center of rotation for edge CD coincides with vertex D. Therefore D is immobile and angle CAD neither increases nor decreases. In fact edge AD is in a state of transition between counter-clockwise and clockwise rotation. However, angle CAB behaves as in case-1 and therefore angle BAD and diagonal BD increase. In this case two edges, AB and CB, rotate clockwise, edge DC rotates counter-clockwise about its end point D, and edge AD is immobile.

4.2.3 Case-3: D lies below L_C and above the x-axis

The case when D lies below L_C and above the x-axis is illustrated in Figure 7. Recall that A lies at the origin of the plane. The diad ABC is as in case-1 and so as AC increases, B rotates clockwise as before and the angle BAC increases. On the other hand O_{CD}, the center of instantaneous rotation for D, is now located left of D and therefore D now also rotates clockwise. Since now angle DAC decreases, it is not clear from the two internal angles at A what angle DAB is doing. However, note that edge CD rotates counter-clockwise about O_{CD} whereas edge CB rotates clockwise about O_{CB}. Therefore the external angle at C increases and so does the diagonal BD and the internal

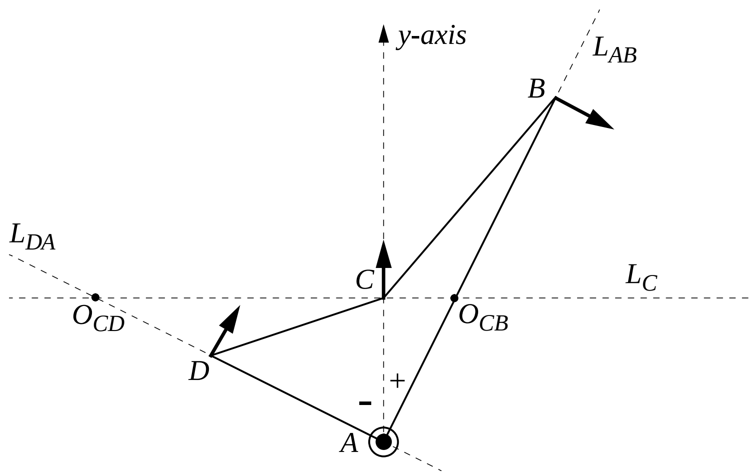

Figure 7: Illustrating the proof for the *line-tracking* motion for Case-3.

angle BAD. Note that in this case only edge CD rotates counter-clockwise. The other three edges rotate clockwise.

4.2.4 Case-4: D lies on the x-axis

Consider now the case when D lies on the x-axis and refer to Figure 8. Now there is no instantaneous center of rotation for edge CD. More precisely, the center is at infinity and therefore CD undergoes a translation as AC increases. Since point C translates upwards, so does D. Although edge CD only translates, edge CB rotates clockwise and therefore the external angle at C increases and so does the diagonal BD and the internal angle BAD. In this case edge CD translates upwards and the other three edges rotate clockwise.

4.2.5 Case-5: D lies strictly below the x-axis

Finally, consider the case when D lies strictly below the x-axis and refer to Figure 9. Now O_{CD}, the instantaneous center of rotation for edge CD lies to the right of C and therefore D rotates clockwise with respect to A. Therefore angle DAC decreases. Hence the two internal angles at A change in oposite directions. Furthermore, the edges CD and CB are both rotating in the same clockwise direction. This case is more difficult than the others because

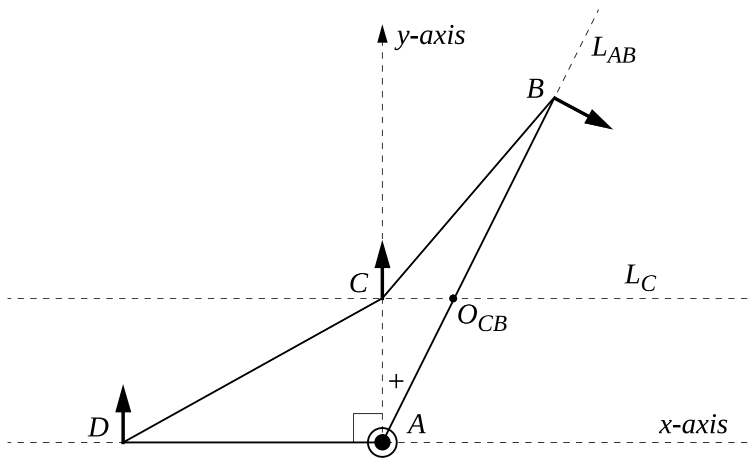

Figure 8: Illustrating the proof for the *line-tracking* motion for Case-4.

all four edges are rotating clockwise making it not obvious what happens to the external angle at C. We will show that edge BC rotates *faster* than edge CD thus proving that the external angle at C and the diagonal BD increase.

Note that the relative motions of the edges with respect to each other are preserved if instead of fixing A and translating C upwards, we fix C and translate A downwards. Therefore let us cut the linkage into two diads at A and C, rotate the diad CDA by an angle of π, and reconnect it to diad ABC to form the new linkage $AD'CB$ (refer to Figure 10). Now when C moves upwards the new edge CD' rotates at the same speed and in the same clockwise direction as the old edge AD. First we show that CB rotates faster than AD by comparing CB with CD'. Since in the original linkage $ABCD$, vertex C is concave, it follows that the new linkage $AD'CB$ is a crossing linkage. By the results for crossing linkages, if AC increases then BD' decreases, thus decreasing angle BCD'. Therefore, although both edges CB and CD' rotate clockwise, edge CB rotates faster than edge CD'. Therefore edge CB rotates faster than edge AD in the original linkage. Finally, note that since both edges AD and DC rotate clockwise in the original linkage, and as AC increases so does angle ADC, it follows that AD rotates faster than DC. Therefore edge BC rotates faster than CD. We conclude that the external angle BCD and diagonal BD increase.

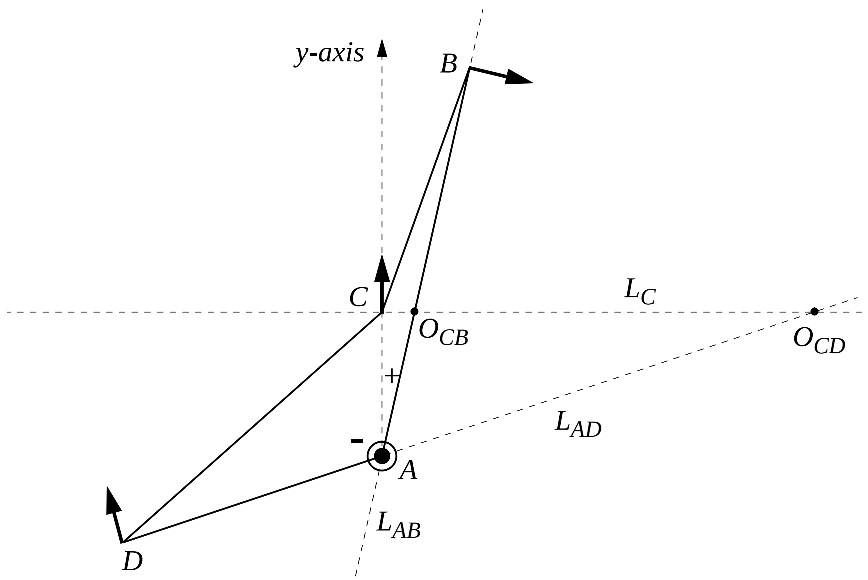

Figure 9: Illustrating the proof for the *line-tracking* motion for Case-5.

4.3 Third proof - via six-bar linkage construction

Case-5 in the previous proof suggests an elegant proof that has no cases and no need for fixing either an edge or a vertex as in the previous two proofs. It works just as well for "floating" linkages.

Consider then a concave linkage $ABCD$ as before with C as the concave vertex. Instead of cutting and reconnecting diad ADC, as was done in Case-5 of the previous proof, we add a new diad AEC to obtain a six-bar linkage such as that illustrated in Figure 11. This new diad lies on the same side of line L_{AC} as B does, and the lengths of its edges are chosen so that together with D it forms a parallelogram $AECD$. From the same arguments used in Case-5 of the previous proof it follows that the new diad forms with B a four-bar crossing sub-linkage $ABCE$. So the six-bar linkage contains all three types of four-bar sub-linkages (convex, concave non-crossing and crossing). Consider now what happens to the three four-bar sub-linkages in the six-bar linkage when AC increases. In the crossing linkage $ABCE$ we have shown that EB decreases and thus so does the acute angle BCE. In the convex linkage $AECD$ we have shown that DE decreases and thus so does the acute angle DCE. But these two angles together with the external angle BCD

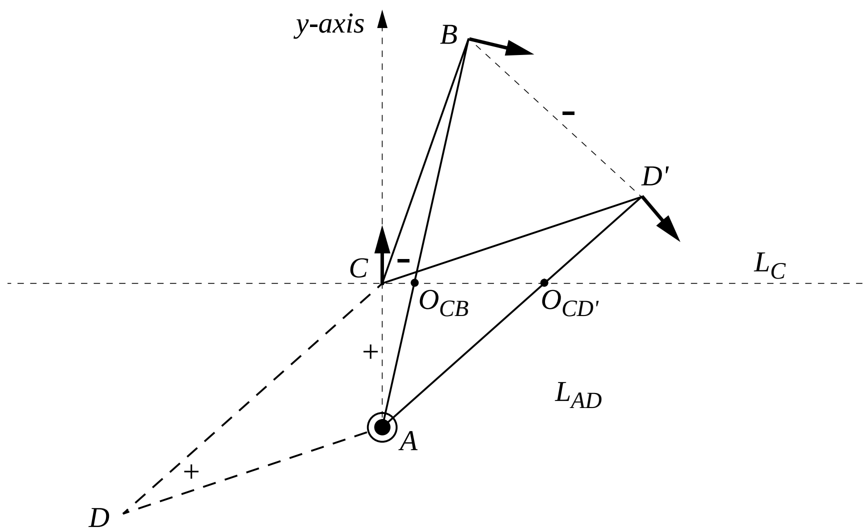

Figure 10: Proving that CB rotates faster than CD' in Case-5.

sum to 2π. Therefore the external angle BCD and diagonal BD increase.

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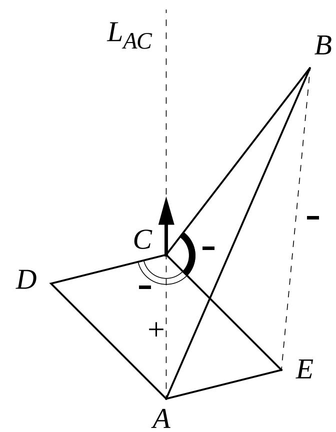

Figure 11: Illustrating the third proof using the six-bar linkage construction.

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