Characterizations of Convex and Star-Shaped Polygons

Thomas Shermer
Godfried Toussaint

School of Computer Science
McGill University
Montreal, Quebec, Canada

ABSTRACT

A *chord* of a simple polygon P is a line segment [x,y] that intersects the boundary of P only at both endpoints x and y. A chord of P is called an *interior* chord provided the interior of [x,y] lies in the interior of P. P is said to be *weakly visible* from [x,y] if for every point v in P there exists a point w in [x,y] such that [v,w] lies in P. In this paper we characterize star-shaped and convex polygons in terms of weak visibility properties of their internal chords. We also provide a new Krasnoselskii-type characterization of *isothetic* star-shaped polygons

1. Introduction

This paper is concerned primarily with new characterizations of star-shaped polygons. As a corollary we also obtain a new characterization of convex polygons.

For any integer $n \geq 3$, we define a *polygon* in the Euclidean plane E^2 as the figure $P = [x_1,x_2,...,x_n]$ formed by n points $x_1,x_2,...,x_n$ in E^2 and n line segments $[x_i,x_{i+1}]$, i=1,2,...,n-1, and $[x_n,x_1]$. The points x_i are called the *vertices* of the *polygon* and the line segments are termed its *edges*. We assume the vertices of P are in *general position*, i.e., no three vertices are collinear.

Definition: A polygon P is called a *simple* polygon provided that no point of the plane belongs to more than two edges of P and the only points of the plane that belong to precisely two edges are the vertices of P.

A simple polygon has a well defined interior (denoted by *int*(P)) and exterior (denoted by *ext*(P)). We will follow the convention of including the interior of a polygon when referring to P. The vertices of a simple polygon are of two types: *convex* and *concave*. In the mathematics literature the terminologies *reentrant vertex* or *local non-convexity point* are often used for *concave vertex* whereas in the computational geometry literature the word *reflex* vertex is preferred. However we shall use the more natural term *concave*. For a given vertex x_j let $y = \lambda x_{j-1} + (1-\lambda)x_j$ and $z = \mu x_{j+1} + (1-\mu)x_j$. For all sufficiently small positive values of μ and λ we have that *int*[y,z] lies either totally in *int*(P) or wholly in *ext*(P); in the former case x_j is a *convex* vertex whereas in the latter case it is a *concave* vertex.

Definition: A simple polygon P is called *star-shaped* if there exists a point x in P such that for all points y in P, [x,y] lies in P. The collection of all such points x is called the *kernel* of P.

Definition: (Grunbaum [Gr], 1975) A simple polygon P is called *convex* provided that all its vertices are convex. Actually this is a special case of a well known theorem due to Tietze [Ti] which states that if S is a closed connected set in a Euclidean space all of whose points are points of local convexity, then S is convex. A point $x \in S$ is a point of local convexity of S if there exists a neighborhood N of x such that $N \cap S$ is convex; otherwise x is called a point of local non-convexity of S.

Characterizations of objects such as convex and star-shaped polygons as well as more general sets are of interest for at least two reasons. Mathematicians are interested in broadening the understanding of geometric objects such as polygons. Different characterizations of an object provide different views of the object and thus further this understanding [Fa],[Kl],[SV],[Va]. Computer scientists on the other hand are interested in designing algorithms for recognizing such objects. Different characterizations yield different algorithms with different complexities for solving such problems [ATB],[BT],[To].

A simple polygon P is said to be convex if every pair of points x,y in P can be joined by a line segment [x,y] that lies totally in P. This very well known characterization of convex polygons is equivalent to the demand that all three of the segments determined by each triplet of pairwise distinct points in P lie totally in P. One can relax this criterion and still obtain a characterization of convex polygons. A simple polygon P is convex iff it contains two of the three segments determined by each triple of its points [MS]. Further weakening the criterion to the new demand that only one of the three segments be contained in P does not lead to convexity but to the notion of P_3-convexity [Va1]. Valentine [Va1] has shown that a P_3-convex polygon can be represented as the union of three or fewer convex polygons.

Convex polygons have also been characterized in terms of nearest point properties [Va3], as illustrated by the *Theorem of Bunt-Motzkin* [YB] which states that a simple polygon P is convex iff for every point **p** not belonging to P there is exactly one point of P nearest to **p**.

There has also been interest in attempting to characterize convex polygons in terms of *unimodality* properties. There are several possibilities for definitions of the notion of unimodality depending on the distance functions employed. For example, one can define for a vertex **z** of P a function f(**z**) which is the Euclidean distance between **z** and each vertex of P in the order in which the vertices occur in P. If f(**z**) is unimodal then we call **z** a unimodal vertex. It was incorrectly assumed for some time that a polygon was convex if all its vertices were unimodal in this sense. Furthermore algorithms for computing geometric structures based on this assumption have been published. However, counter examples to the claim [ATB] and to such algorithms [BT] have since appeared. Just as we measured Euclidean distance between pairs of vertices to create f(**z**) we can instead consider vertex-edge or edge-vertex pairs and measure the separation as the perpendicular distance between the vertex and the line collinear with the edge in question. In this way for an edge **e** of P we can define a function g(**e**) which is the perpendicular distance from the line collinear with **e** to every

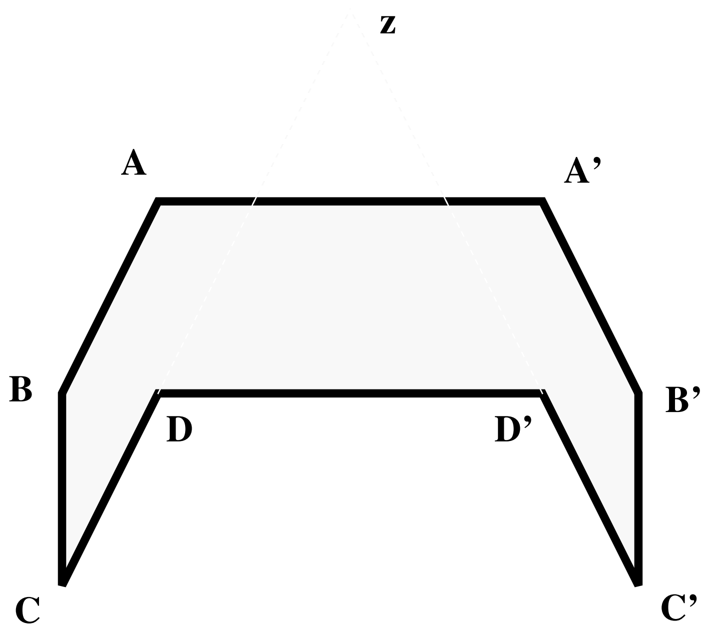

Fig. 1.1

vertex of P in the order the vertices appear in P. If g(**e**) is unimodal we call **e** a unimodal edge. In [To] it is shown that if P has all its edges unimodal in this sense then P is convex.

A very different type of unimodality to that discussed above was considered by Dharmadhikari and Jogdeo [DJ]. Let P be a simple polygon in R^2. For a fixed non-zero vector $u \in R^2$ and $k \in R$, let L(u,k) denote the line **u.** x = k and let $\phi_u(k)$ denote the measure of $P \cap L(u,k)$. If P is a convex polygon then $\phi_u(k)$, as a function of k, is first non-decreasing and then non-increasing for every value of u. A non-negative function f on R is said to be *unimodal* if there is a $v \in R$ such that f is non-decreasing on $(-\infty,v]$ and non-increasing on $[v,\infty)$. Furthermore such a number v need not be unique. Consider now the following condition:

Condition A: For every fixed non-zero $u \in R^2$, the function $\phi_u(k)$ is unimodal in k.

It is natural to ask whether condition A is sufficient for a simple polygon P to be convex. The answer to this question is negative. Consider the following example found in [DJ] and refer to Fig. 1.1. Let ABCD and A'B'C'D' be parallelograms which are mirror images of each other and are such that CD and C'D' meet outside the polygon at some point z. It can easily be verified that this polygon satisfies condition A but it is not even star-shaped. On the other hand in [DJ] it is shown that if P is such that for every fixed $u \in R^2$, the function $\phi_u(k)$ is *continuous* on the interior of its support then P is convex.

The earliest characterization of star-shaped polygons is due to Krasnoselskii [Kr] and this result has become known as Krasnoselskii's theorem [YB]. This theorem states that if for every set of three points $x,y,z \in P$ there exists a point $w \in P$ (possibly dependent on x,y,z) such that the three segments [w,x], [w,y], [w,z] all lie in P then P is star-shaped. If a polygon is *isothetic* then it has been shown that the number *three* can be reduced to *two* [TE]. A polygon P is called *isothetic* provided that all its edges are parallel to either the x or y axes. For additional Krasnoselskii-type characterizations of star-shaped polygons the reader is referred to [Br],[Mo], [Ro], and [Va2].

2. Weak Visibility Characterizations

In this section we present new characterizations of star-shaped and convex polygons based on the notion of weak visibility. A polygon P is said to be *weakly visible* [AT] from a subset S of P if for every point x in P there exists a point y in S such that the line segment [x,y] lies in P. A *chord* of a polygon P is a line segment [x,y] that intersects the boundary *bd*(P) only at x and y. If the interior of [x,y] lies in the interior of P then we say the chord is an *interior chord*. If the interior of [x,y] lies in the exterior of P then we say the chord is an *exterior chord*.

Theorem 2.1: A simple polygon P is star-shaped iff there exists a point $x \in P$ such that P is weakly visible from every internal chord traversing x.

Proof: (only if) Choose x to be any point in the kernel of P. Since P is star-shaped from x it follows that it is weakly visible from any internal chord of P that traverses x.

(if) Assume P contains a point x such that P is weakly visible from every internal chord traversing x. We claim that P is in fact star-shaped from x. If this were not so it would imply the existence of a point y in P that is not visible from x. Now construct a line through both x and y and let a,b denote the first points of intersection of L with *bd*(P) as we traverse L in both directions starting at x. Let $L' \in L$ denote the segment [a,b]. Since y, a and b all lie on L and y is not visible from x it follows that y is not visible from any point on L'. Therefore P is not weakly visible from chord L' which is a contradiction. Q.E.D.

Note that as a corollary we obtain a new characterization of convex polygons. The theorem actually proves that if a point x exists such that P is weakly visible from every internal chord traversing x then P is star-shaped from x. Thus if this property holds true for every point $x \in P$ it follows that P is star-shaped from with respect to every point in P and is thus convex. We thus have the following result.

Corollary 2.1: A simple polygon P is convex iff it is weakly visible from every internal chord of P.

Consider now the case of isothetic polygons, i.e., polygons with all their edges parallel to the coordinate axes. Such a polygon with its four types of *tabs* is illustrated in Fig. 2.1. A *tab* is a set of two adjacent convex vertices along with the three edges of P incident on these two vertices. There are four types of tabs. For example, in Fig. 2.1 [a,b,c,d] is a *top tab*. Recall that theorem 2.1 states that an arbitrary simple polygon P is star-shaped iff there exists a point $x \in P$ such that P is weakly visible from *every* internal chord traversing x. The word *every* is highlighted to indicate that for every (i.e., an infinite number) unoriented direction θ there exists such a chord. The term *unoriented direction* θ refers to an equivalence class of parallel lines that make an angle of θ with respect to some agreed upon fixed axis. Also observe that in an arbitrary simple polygon P each of its edges can occur in any one of an infinite number of unoriented directions. Return now to the case of isothetic polygons. It is natural to conjecture the following result analogous to theorem 2.1.

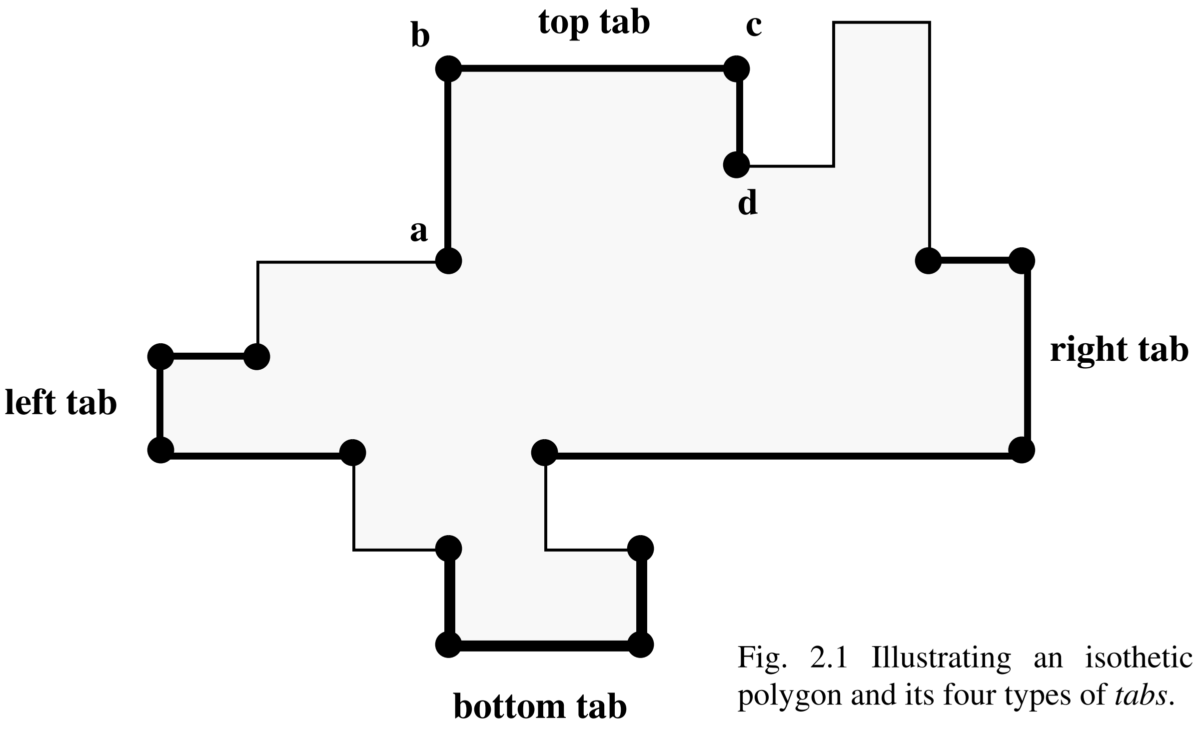

Fig. 2.1 Illustrating an isothetic polygon and its four types of *tabs*.

Conjecture: A simple *isothetic* polygon P is star-shaped iff there exists a point $x \in P$ such that P is weakly visible from both the *horizontal* and *vertical* internal chords traversing x.

As it turns out however we can prove a stronger result for isothetic polygons in the form of the following theorem.

Theorem 2.2: A simple *isothetic* polygon P is star-shaped iff P is weakly visible from both some *horizontal* and some *vertical* internal chord of P.

Proof: (only if) Let x be a point in the kernel of P. Clearly P is weakly visible from any internal chord traversing point x. Therefore P is weakly visible from both some horizontal and some vertical internal chord of P, namely the horizontal and vertical chords traversing point x.

(if) Let P be weakly visible from some vertical internal chord $[t,b]$ where t and b are the upper and lower endpoints, respectively, of the chord. It follows that t must occur on a top tab and b on a bottom tab for otherwise there would exist at least one vertex of P not visible from $[t,b]$. Now $[t,b]$ decomposes P into two polygons P_1 and P_2. Furthermore, polygons P_1 and P_2 cannot themselves contain any top or bottom tabs other than those determined by $[t,b]$ or they would contain vertices not visible from $[t,b]$. Therefore P must contain only one top tab and only one bottom tab and $[t,b]$ must connect these two tabs. Similar arguments show that P must contain precisely one left tab and one right tab and that $[l,r]$, the horizontal chord from which P is weakly visible must have its endpoints l,r on the unique left and right tabs, respectively. Furthermore, the right tab of P must lie to

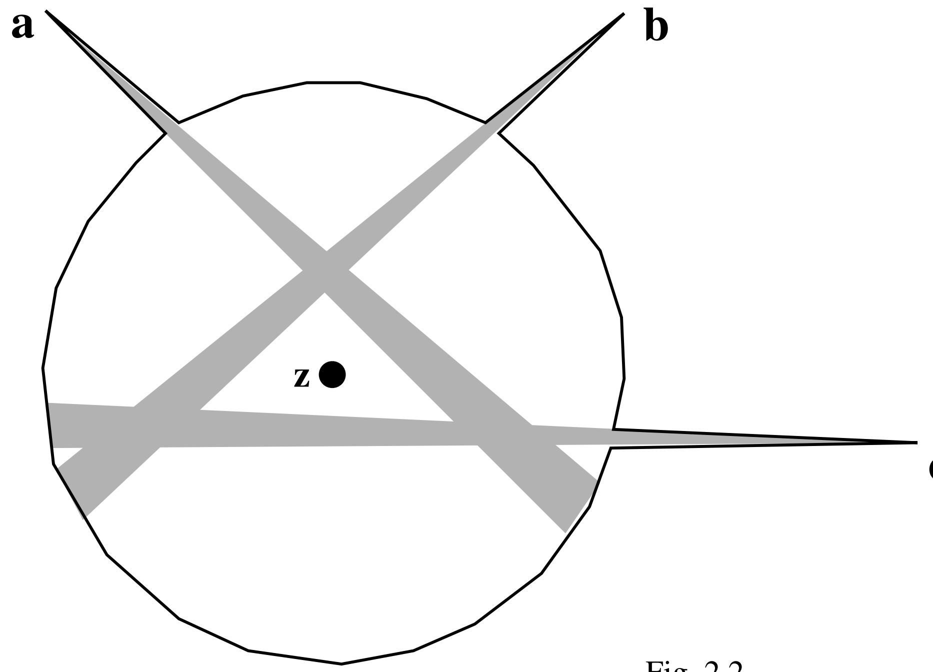

Fig. 2.2

the right of [*t,b*] and the left tab of P must lie to the left of [*t,b*]. For assume this not to be the case and, without loss of generality, let P contain a right tab to the left of [t,b]. This would imply that P_1 or P_2 contains a top or bottom tab other than those determined by [t,b] which in turn would contradict the fact that P is weakly visible from [t,b]. Therefore each horizontal chord that weakly sees P will intersect each such vertical chord. It still remains to show that P is star-shaped. Let z be the intersection point of [*t,b*] and [*l,r*]. We will show that P is star-shaped from *z*. Assume P is not star-shaped from *z*. This implies that there exists a point *w* on *bd*(P) that is not visible from *z*. Without loss of generality assume that *w* lies on that part of *bd*(P) between *r* and *b* as P is traversed in a clockwise manner and denote this portion of P by Chain[r,...,b]. Let P_{zrb}= Chain[*r*,...,*b*] $\cup$ [*b,z*] $\cup$ [*z,r*] and let VP[P_{zrb},*z*] be the visibility region of P_{zrb} from *z*. VP[P_{zrb},*z*] cuts off regions of P_{zrb} which are hidden from *z* and are either to the left or to the right of the cutting visibility rays emanating from *z*. Clearly *w* must lie either in a left or a right such hidden region. In the former case *w* is not visible from [*l,r*] and in the latter case *w* is not visible from [*t,b*]. In both cases we have a contradiction and therefore P must be star-shaped from *z*. Q.E.D.

We have shown in theorem 2.2 that the point x in the conjecture could be disposed of and that it was sufficient to impose weak visibility from some horizontal and some vertical chord in order to characterize isothetic star-shaped polygons. This opens a similar question for the original non-isothetic simple polygons. In other words, is it true that an arbitrary simple polygon P is star-shaped iff for every unoriented direction θ there exists an internal chord of P with direction θ from which P is weakly visible? The answer to this question is negative and a counterexample due to ElGindy [El] is illustrated in Fig. 2.2. Apart from three thin spikes at **a**, **b**, and **c** the polygon in Fig. 2.2 has its remaining vertices on a circle with center **z**. Furthermore the spikes are so thin and so

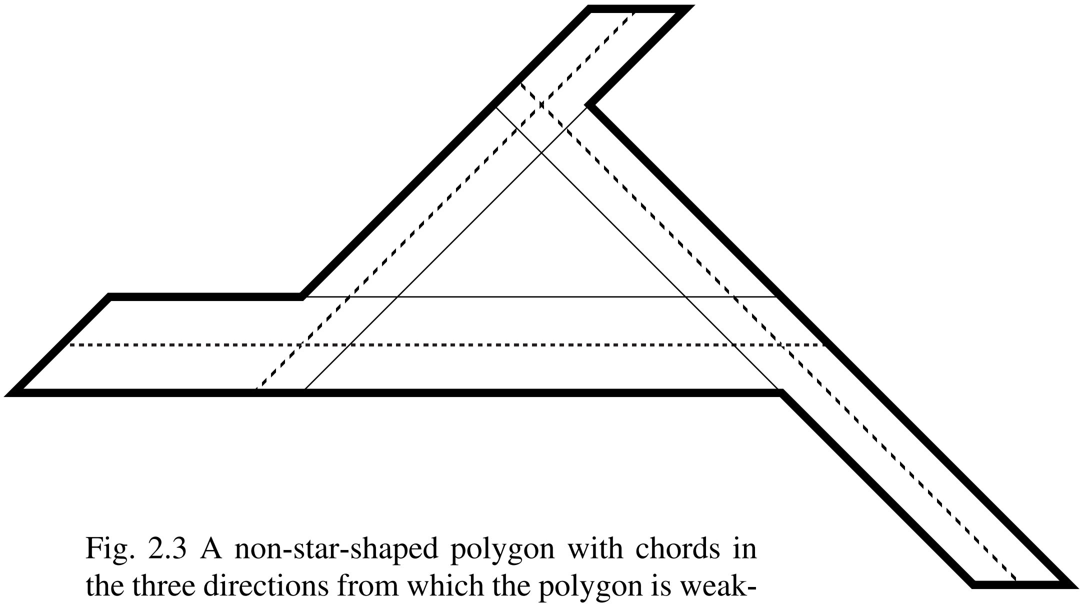

Fig. 2.3 A non-star-shaped polygon with chords in the three directions from which the polygon is weakly visible.

placed that their extended visibility lines form a triangle that encloses **z**. It is clear by observation that for every unoriented direction θ there exists an internal chord of P with direction θ from which P is weakly visible. Consider any chord of P with direction θ and passing through **z**. If θ is not contained in the set of directions determined by the three spikes then P is weakly visible from this chord. On the other hand if θ is contained in one such set, say that of spike **a**, then translate the chord in a direction orthogonal to θ until it intersects the visibility cone of spike **a**. However, note that P is not star-shaped. Therefore theorem 2.2 which concerns polygons with edges parallel to two directions does not have its counterpart in the case of polygons with edges parallel to an infinite number of directions. An obvious question arises. Does theorem 2.2 have a counterpart for a finite fixed number of directions. In other words, if P is such that all its edges are parallel to k fixed unoriented directions, where k is some fixed positive integer, is it true that P is star-shaped iff P is weakly visible from some chord in each of the k directions? The answer to this question is also negative and a counterexample for the case of three directions is shown in Fig. 2.3. This polygon is weakly visible from each of the three dotted lines parallel to the three directions constraining the edges of P and yet it is not star-shaped.

3. Krasnoselskii-type Characterizations of Isothetic Star-shaped Polygons

In this section we present two characterizations of isothetic star-shaped polygons which resemble Krasnoselskii's theorem [Kr] for arbitrary simple polygons in their combinatorial flavor. First we state the original theorems for arbitrary and isothetic polygons.

Theorem 3.1: (Krasnoselskii [Kr], 1946) If every *three* points on the boundary of a simple polygon P are visible from some common point in P then there exists a point in P from which the entire boundary of P is visible.

Theorem 3.2: (Toussaint & ElGindy [TE], 1981) If every *two* points on the boundary of an *isothetic* simple polygon P are visible from some common point in P then there exists a point in P from which the entire boundary of P is visible.

In order to proceed straightforwardly we introduce some definitions for arbitrary simple polygons.

Definition: The two closed rays of a line L which have only a point $x \in L$ in common are called *complementary* rays. If R(x) is a ray with endpoint x, its complementary ray is denoted by R'(x).

Definition: A set of rays is said to be *concurrent* if there exists a point of the plane that intersects each and every ray in the set.

Definition: A ray R(x) with endpoint $x \in bd(P)$ is an *external ray of support to int*(P) if $R(x) \cap int(P) = \emptyset$.

Definition: If $x \in bd(P)$ then K(x) is the union of all the external rays of support to *int*(P) at x. The set K(x) is called an *external cone of support*. The union of all the complementary rays R'(x) where $R(x) \subset K(x)$ is denoted by K'(x).

Valentine [Va2] proved the following result.

Theorem 3.3: (Valentine[Va2], 1965) Let P be a non-convex simple polygon. Suppose that for each set of *three* (not necessarily distinct) concave vertices x_1,x_2,x_3 of P, there exist three external rays of support at x_1,x_2,x_3 respectively to *int*(P) whose corresponding complementary rays are concurrent and meet in P. Then P is star-shaped.

We now prove the main result of this section.

Theorem 3.4: Let P be a non-convex simple *isothetic* polygon. Suppose that for each set of *two* (not necessarily distinct) *concave* vertices x_1,x_2 of P, there exist two external rays of support at x_1,x_2 respectively to *int*(P) whose corresponding complementary rays are concurrent and meet in P. Then P is star-shaped.

Proof: Let x_1 and x_2 be two concave vertices of P. By hypothesis there exist two external rays of support at x_1,x_2 respectively to *int*(P) whose corresponding complementary rays are concurrent and meet in P at some point z. By construction it follows that $[x_1,z] \in K'(x_1)$ and $[x_2,z] \in K'(x_2)$. Therefore $z \in [K'(x_1) \cap K'(x_2)]$. Since P is isothetic $K'(x_1)$ and $K'(x_2)$ can each be expressed as the intersection of the half-planes (containing z) determined by $bd[K'(x_1)]$, and $bd[K'(x_2)]$. Denote the vertical and horizontal such half-planes by $V(x_i)$ and $H(x_i)$, respectively, where i=1,2. Clearly the pair $V(x_1)$ and $V(x_2)$ must contain a non-zero intersection. Since the above arguments are true for all pairs of concave vertices of P it follows that all pairs of vertical half-planes intersect. It follows from Helly's theorem [He] that all such vertical half-planes contain a non-zero intersection. Similar arguments hold for the horizontal half-planes. Therefore the intersection of K'(x_i) over all concave vertices x_i must be non-zero. Denote this intersection by K*. Next we will show that the intersection of K* with P is non-zero. First note that P cannot contain more than one of each of the four types of tabs. For if this were not so one type would contain at least two tabs and this would imply that there exist at least two concave vertices with external rays of support whose corresponding complementary rays are not concurrent contradicting the hypothesis. Let K** denote the intersection of K* with the four interior half-planes determined by each of the four tabs. Each such half-plane contains P and is bounded by the line collinear with the two convex vertices making up the corresponding tab. It is a straight forward matter to show that if K* is bounded this intersection operation will not change K* and if it is unbounded then K** will be bounded but non-zero. Note also that the above discussion implies that each convex vertex of P not belonging to a tab must be such that both of its adjacent vertices are concave. This in turn implies that K** is the intersection of the interior half-planes determined by all the edges of P and therefore P has a non-zero kernel. Therefore P is star-shaped.   Q.E.D.

References

[AT] Avis, A. and Toussaint, G. T., "An optimal algorithm for determining the visibility of a polygon from an edge," *IEEE Transactions on Computers,* vol. C-30, No. 12, December 1981, pp.910-914.

[ATB] Avis, D., Toussaint, G.T., and Bhattacharya, B.K., "On the multimodality of distances in convex polygons," *Computers & Mathematics With Applications*, vol. 8, No. 2, 1982, pp. 153-156.

[Br] Breen, M., "Krasnoselskii-type theorems," in *Discrete Geometry and Convexity*, Eds., J. E. Goodman et al., New York Academy of Sciences, 1985, pp.142-146.

[BT] Bhattacharya, B.K. and Toussaint, G.T., "A counterexample to a diameter algorithm for convex polygons," *IEEE Trans. Pattern Analysis and Machine Intelligence,* vol. PAMI-4, No. 3, May 1982, pp. 306-309.

[DJ] Dharmadhikari, S. W. and Jogdeo, K., "A characterization of convexity and central symmetry for planar polygonal sets," *Israel Journal of Mathematics*, vol. 15, 1973, pp.356-366.

[El] ElGindy, H., personal communication.

[Fa] Fary, I., "A characterization of convex bodies," *American Mathematical Monthly*, vol. 69, 1962, pp.25-31.

[Gr] Grunbaum, B., "Polygons," in *The Geometry of Metric and Linear Spaces*, A. Dold & B. Eckman, eds., Springer-Verlag, New York, 1965, pp.147-184.

[He] Helly, E., "Uber Mengen konvexer Korper mit gemeinshaftlichen Punkten," *Jber. Deutsch. Math. Verein*. vol. 32, 1923, 175-176.

[Kl] Klee, V. L., "A characterization of convex sets," *American Mathematical Monthly*, vol. 56, 1949, pp.247-249.

[Kr] Krasnoselskii, M. A., "Sur un critere qu'un domaine soit etoile, *Math. Sb.* (61)19(1946).

[MS] Marr, J. M. and Stamey, W. L., "A three-point property," *American Mathematical Monthly*, vol. 69, 1962, pp.22-25.

[Mo] Molnar, J., Uber Sternpolygone, *Publ. Math. Debrecen.* vol. 5, 1958, pp.241-245.

[Ro] Robkin, E. E., *Characterizations of star-shaped sets*, Ph.D. thesis, UCLA, 1965.

[SV] Strauss, E. G. and Valentine, F. A., "A characterization of finite dimensional convex sets," *American Journal of Mathematics,* vol. 74, 1952, pp.683-686.

[Ti] Tietze, H., "Uber Konvexheit im kleinen und im grossen und uber gewisse den Punkten einer Menge zugeordnete Dimensionzahlen, *Math. Z.* vol. 28, 1928, pp.697-707.

[To] Toussaint, G. T., "Complexity, convexity, and unimodality," *International Journal of Computer and Information Sciences*, vol. 13, No. 3, June 1984, pp.197-217.

[TE] Toussaint, G. T. and ElGindy, H., "Traditional galleries are star-shaped if every two paintings are visible from some common point," Tech. Rept. SOCS-81.10, School of Computer Science, McGill University, March 1981.

[Va1] Valentine, F. A., "A three point convexity property," *Pacific Journal of Mathematics*, vol. 7, 1957, pp.1227-1235.

[Va2] Valentine, F. A., "Local convexity and star-shaped sets," *Israel Journal of Mathematics*, vol. 3, March 1965, pp.39-42.

[Va3] Valentine, F. A., *Convex Sets*, McGraw-Hill, New York, 1964.

[Va4] Valentine, F. A.,"Characterizations of convex sets by local support properties," *Proc. Amer. Math. Soc.,*vol. 11, 1960, pp.112-116.

[YB] Yaglom, I. M. and Boltyanskii, V. G., *Convex Figures*, Holt, Rinehart and Winston, New York, 1961.