A Note on Linear Expected Time Algorithms for Finding Convex Hulls

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Abstract

Consider n independent identically distributed random vectors from $\mathbf{R}^d$ with common density f, and let $E(C)$ be the average complexity of an algorithm that finds the convex hull of these points. Most well-known algorithms satisfy $E(C) = O(n)$ for certain classes of densities. In this note, we show that $E(C) = O(n)$ for algorithms that use a "throw-away" pre-processing step when f is bounded away from 0 and ∞ on any nondegenerate rectangle of $\mathbf{R}^2$.

1 Introduction

Let $X_1, \ldots, X_n$ be independent identically distributed random vectors from $\mathbf{R}^d$ with common density f, and let C be the complexity of a given convex hull algorithms for $X_1, \ldots, X_n$ (thus, C is a random variable). In this note we will discuss several convex hull algorithms and the condition on f that will insure their linear average time behavior:

$$E(C) = O(n) \tag{1}$$

In general, the more sophisticated algorithms satisfy (1) for a larger class of densities than do the simple algorithms. The purpose of this note is merely to draw the attention to a particularly simple algorithm and prove that it satisfies (1) for a small but frequently encountered class of densities. We will first review some well-known convex hull algorithms and indicate the densities for which (1) holds. We will assume that $d = 2$.

1. Graham's algorithm [12] sorts the X_i's according to the angles between the x-axis and the lines joining the X_i's with an interior point. Then it finds the convex hull in time $O(n)$. If bucket sorting is used on the angles, then (1) holds whenever f is bounded and has compact support [8].

2. Bentley and Shamos [3] showed that their "divide-and-conquer" algorithm satisfies (1) whenever f is such that $E(N_c) = O(n^p)$ for some $p < 1$ where N_c is the number of X_i's on the convex hull. Most well-known densities satisfy thier condition.

3. Jarvis' simple algorithm [13] has $E(C) = O(n)$ whenever $E(N_c) = O(1)$. In an interesting paper by Carnal [5] it was pointed out that many heavy-tailed radial densities fall into this category. It suffices that for some origin x_0, $X_1 - x_0$ has a radially symmetric distribution such that for all $0 < c < 1$,

$$\lim_{r\to\infty} \frac{P\left(\|X_1 - x_0\| \geq cr\right)}{P\left(\|X_1 - x_0\| \geq r\right)} = \frac{1}{c^\alpha}$$

   for some constant $\alpha \geq 0$. For example, it suffices that

$$f(x) = \text{constant}/\left(\|x\|^{2+\delta} + 1\right),\ x \in \mathbf{R}^2,$$

   for some $\delta > 0$, where $\|x\|$ denotes the standard euclidean norm in $\mathbf{R}^2$.

4. In an indirect approach, one could first find the maximal vectors among $X_1, \ldots, X_n$ and then extract the convex hull from these points using a polynomial time worst-case algorithm. Consider the first quadrant on the plane. A vector X is said to dominate another vector Y is X is greater than Y in both coordinates, i.e., $x_1 > y_1$ and $x_2 > y_2$. A vector X_i is a maximal vector among a set if it is not dominated by any other vector in the set. Analogous definitions hold for the other three quadrants with the corresponding sign changes. The set of maximal vectors forms a superset of the convex hull vectors. If the first part is executed by using the algorithm of Bentley et al. [2], then $E(C) = O(n)$ whenever all the components of X_1 are independent (i.e., f is the product of its marginal densities). This result remains true for $d > 2$. See [7].

5. In another indirect approach Shamos [14] proposed obtaining the convex hull by first computing the Voronoi diagram of the set of points. The Voronoi diagram of a set of points $X_1, \ldots, X_n$ is a partition of the plane into n regions or tiles T_i such that for any X in T_i, X is closer to X_i than to any other vector in the set. Such a partition consists of bounded an unbounded regions. The unbounded regions identify the convex hull points and can be obtained in $O(n)$ time once the Voronoi diagram is computed. Bentley, Weide and Yao [4] showed that when f is such that it has a convex compact support and there exist two positive constants M and m such that $M > f > m$, the Voronoi diagram can be computed in $O(n)$ expected time. It follows that under these conditions Shamos' convex hull algorithm runs in linear expected time.

The algorithm discussed in this note is very sinple but extremely fast and useful (see [1]). In a first step, many points are excluded from further consideration in time $O(n)$. The convex hull of the remaining points is then found by using any of the established convex hull algorithms. More formally, we will consider all algorithms of the following form.

Step 1: Find $X_1^*, \ldots, X_8^*$ from $X_1, \ldots, X_n$ where the X_i^*'s are the extrema (i.e. the points furthest apart) in the $\pm x$, $\pm y$, $\pm(x+y)$, $\pm(x-y)$ directions. Some of the X_i^*'s may coincide. Step 1 takes time $O(n)$.
Step 2: Eliminate from $X_1, \ldots, X_n$ all points that do not belong to the convex polygon P formed by the X_i^*'s.
Step 3: Apply *any* $O(n^2)$ worst-case convex hull algorithm to the points not eliminated in step 2. One should note here that all the algorithms discussed in this note can be used in step 3.

This simple algorithm cannot be expected to satisfy (1) for all densities f. When f is uniform on the unit circle, then on the average $O(n)$ points will be left after steps 1 and 2, and much depends upon the algorithm used in step 3. We do not wish to specify an algorithm in step 3 because steps 1 and 2 should be considered as preprocessing steps in all generality.
Remark 1: Eddy [10] has proposed an algorithm that uses an idea similar to that of steps 1 and 2 but it repeats these steps by finding extrema in different directions instead of proceeding to step 3. Furthermore, after having initially found the X_i^*'s in the x direction (X^*_{min} and X^*_{max}) they search for two points furthest away from and orthogonal to the line through X^*_{min} and X^*_{max}. This seems to require much more computation, in the form of multiplications, than simply finding the extreme points in the directions of step 1 above. Bentley and Shamos [3] mention that Floyd has shown that Eddy's algorithm satisfies (1) for certain symmetric f.

2 The Main Result

Lemma 2.1 *Let $0 < m \leq f(x) \leq M < \infty$ for all x in some nondegenerate rectangle of $\mathbf{R}^2$, and let $f(x) = 0$ elsewhere. Let N be the number of points left after step 2. Then, for all $\varepsilon > 0$,*

$$P(N > \varepsilon) \leq k_1 \exp\left(-\frac{k_2 \varepsilon^2}{n}\right) \tag{2}$$

where k_1 and k_2 are positive constants.

Proof: It suffices to show the lemma when the nondegenerate rectangle is $[0,1]^2$. The proof for the general case is similar but tends to drown the argument in irrelevant details.

Let $(Y_1, Z_1), \ldots, (Y_4, Z_4)$ be the four extrema in the $\pm(x+y)$, $\pm(x-y)$ directions after step 1, and draw four horizontal and four vertical lines through these extrema (see Fig. 1). Let T be the rectangle defined by the innermost lines (all extrema should lie on or outside T). Clearly, $N \leq N_T$ where N_T is the number of X_i's outside T. Let $T_1, \ldots, T_8$ be the eight "strips" between the eight lines and their corresponding parallel edges of the unit square. Also, let $N_1, \ldots, N_8$ be the number of points in these strips, and let $A_1, \ldots, A_8$ be the probabilities of $T_1, \ldots, T_8$.

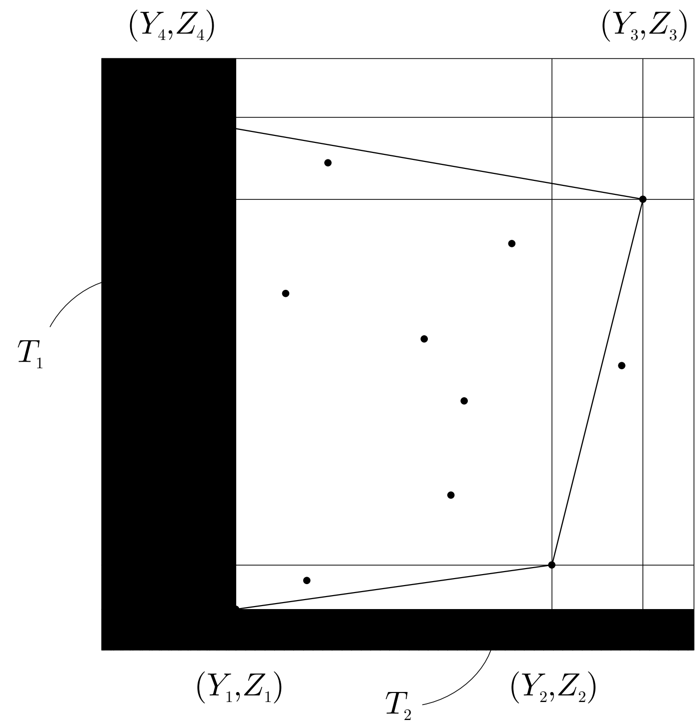

Figure 1: The shaded areas illustrate two of the eight strips $T_i,\ i = 1, \ldots, 8$.

We know that

$$N \leq N_T \leq \sum_{i=1}^{8} N_i,$$

and,

$$\begin{aligned} P(N > \varepsilon) &\leq \sum_{i=1}^{8} P(N_i > \varepsilon/8) \\ &\leq \sum_{i=1}^{8} [P(N_i - nA_i > \varepsilon/16) + P(nA_i > \varepsilon/16)] \end{aligned} \tag{3}$$

Let T_1 be $[0, Y_1] \times [0, 1]$: T_1 is the vertical strip defined by the extremum (Y_1, Z_1) in the $-(x + y)$ direction (closest to the origin). Let F be the distribution function of the x-component of X_1, and let F_n be the empirical distribution function of the x-components of $X_1, \ldots, X_n$, that is,

$$F_n(x) = \frac{1}{n} \times \text{number of } X_i\text{'s with } x\text{-component} \leq x.$$

For any $\varepsilon > 0$,

$$P(N_1 - nA_1 > \varepsilon/16) = P(F_n(Y_1) - F(Y_1) > \varepsilon/16\ n)$$

$$
\begin{aligned}
&\leq P\left(\sup_{0<x<1} F_n(x) - F(x) > \varepsilon/16\ n\right) \\
&\leq k_3 \exp\left(-2n\left(\frac{\varepsilon}{16n}\right)^2\right) \\
&= k_3 \exp\left(-\varepsilon^2/128\ n\right) \qquad (4)
\end{aligned}
$$

Here we use a result due to Dvoretzky, Kiefer, and Wolfowitz [9] (it is known that $k_3 < 611$). Bound (4) is independent of the index i ($1 \leq i \leq 8$).

Finally,

$$
\begin{aligned}
& P(nA_1 > \varepsilon/16) \leq P(MY_1 > \varepsilon/16\ n) \\
\leq\ & P(Z_1 + Y_1 > \varepsilon/16\ nM) \\
=\ & P\left(\text{triangle } (0,0), \left(0, \frac{\varepsilon}{16nM}\right), \left(\frac{\varepsilon}{16nM}, 0\right) \text{ is empty}\right) \\
\leq\ & \left[1 - \frac{1}{2}\left(\frac{\varepsilon}{16nM}\right)^2 m\right]^n \\
\leq\ & \exp\left(-\frac{nm}{2}\left(\frac{\varepsilon}{16nM}\right)^2\right) \\
=\ & \exp\left(-\frac{\varepsilon^2 m}{512nM^2}\right) \qquad (5)
\end{aligned}
$$

Since (5) is valid for all A_i's we have from (3),(4) and (5)

$$
P(N > \varepsilon) \leq 8(k_3 + 1)\exp\left(-\frac{\varepsilon^2 m}{512nM^2}\right),
$$

concluding the proof of the lemma. ■

Theorem 2.2 *When $0 < m \leq f(x) \leq M < \infty$ for some constants m, M on any nondegenerate rectangle in $\mathbf{R}^2$, and $f = 0$ elsewhere, then the elimination algorithm given above satisfies (1), i.e. $E(C) = O(n)$.*

Proof: It is clear that

$$
C \leq k_4 n + k_5 N^2
$$

where k_4 and k_5 are positive constants. Now, (1) follows when $E(N^2) = O(n)$. By a well-known identity (see [11]),

$$
\begin{aligned}
E(N^2) &= \int_0^\infty P(N^2 > t)dt \\
&= \int_0^\infty 2tP(N > t)dt \qquad (6)
\end{aligned}
$$

Substituting (2) into (6) yields

$$\begin{aligned} E(N^2) &\leq 2k_1 \int_0^\infty t \exp\left(-\frac{k_2 t^2}{n}\right) dt \\ &= n\frac{k_1}{k_2}, \end{aligned}$$

thus proving the theorem. ■

Remark 2: The Theorem applies to all elimination algorithms that use 8 equispaced directions in step 1. For some densities, the result of the Theorem can be obtained by using fewer equi-spaced directions. For example, when f is the standard normal density, then 3 equi-spaced directions suffice to conclucde that $E(C) = O(n)$, provided that in step 3 an $O(n \log n)$ worst-case convex hull algorithm is employed ([6]).

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