All meals for a dollar and other vertex enumeration problems

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Diet problem

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- Objective Eat for less than \$1.

Sample data

	Food	Serv. Size	Energy (kcal)	Protein (g)	Calcium (mg)	Price ¢	Max Serv.
x_1	Oatmeal	28g	110	4	2	3	4
x_2	Chicken	100g	205	32	12	24	3
x_3	Eggs	2 large	160	13	54	13	2
x_4	Milk	237ml	160	8	285	9	8
x_5	Cherry Pie	170g	420	4	22	20	2
x_6	Pork w. beans	260g	260	14	80	19	2
	Min. Daily Amt.		2000	55	800		

The decision variables are $x_1, x_2, \ldots, x_6$.

Fractional servings are allowed.

From *Linear Programming*, Vasek Chvátal, 1983

Linear programming formulation for diet problem

	Food	Serv. Size	Energy (kcal)	Protein (g)	Calcium (mg)	Price ¢	Max Serv.
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$$
\begin{array}{rrcl}
 & \min z & = & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\
\text{s.t.} & 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 & \geq & 2000 \\
 & 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 & \geq & 55 \\
 & 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 & \geq & 800 \\
 & \multicolumn{3}{l}{0 \leq x_1 \leq 4, \quad 0 \leq x_2 \leq 3, \quad 0 \leq x_3 \leq 2,} \\
 & \multicolumn{3}{l}{0 \leq x_4 \leq 8, \quad 0 \leq x_5 \leq 2, \quad 0 \leq x_6 \leq 2}
\end{array}
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Linear programming solution

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- $x_1 = 4$(oatmeal) $x_4 = 4.5$(milk) $x_5 = 2$(pie) cost=92.5 ¢

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- $x_1 = 4$(oatmeal) $x_4 = 4.5$(milk) $x_5 = 2$(pie) cost=92.5 ¢
- Where are the chicken, eggs and pork?
- Do I have to eat the same food every day?

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- Ask the right question !
- What are all the meals I can eat for at most $1?

All meals for a dollar

Replace the objective function by an inequality:

$$3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \leq 100$$

$$\begin{aligned}
110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\geq 2000 \\
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0 \leq x_1 \leq 4, \quad 0 \leq x_2 \leq 3, \quad 0 \leq x_3 \leq 2, & \\
0 \leq x_4 \leq 8, \quad 0 \leq x_5 \leq 2, \quad 0 \leq x_6 \leq 2 &
\end{aligned}$$

All meals for a dollar

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- Any solution to these inequalities is a meal for under \$1
- But this is just a restatement of the problem ....
- ... how do I find these solutions?

A more useful solution

All menus for a $1

All (17) Extreme
Solutions to the Diet Problem with Budget $1.00

Cost	Oatmeal	Chicken	Eggs	Milk	Cherry Pie	Pork Beans
92.5	4.	0	0	4.5	2.	0
97.3	4.	0	0	8.	0.67	0
98.6	4.	0	0	2.23	2.	1.40
100.	1.65	0	0	6.12	2.	0
100.	2.81	0	0	8.	0.98	0
100.	3.74	0	0	2.20	2.	1.53
100.	4.	0	0	2.18	1.88	1.62
100.	4.	0	0	2.21	2.	1.48
100.	4.	0	0	5.33	2.	0
100.	4.	0	0	8.	0.42	0.40
100.	4.	0	0	8.	0.80	0
100.	4.	0	0.50	8.	0.48	0
100.	4.	0	1.88	2.63	2.	0
100.	4.	0.17	0	2.27	2.	1.24
100.	4.	0.19	0	8.	0.58	0
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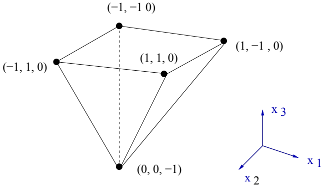
- Taking convex combinations of rows gives new meals
- Eg. Taking half each of the last two rows gives a $1 meal with all foods

10

Example in R^3



H-representation:

$$
\begin{aligned}
1 - x_1 \qquad + x_3 &\geq 0 \\
1 \qquad - x_2 + x_3 &\geq 0 \\
1 + x_1 \qquad + x_3 &\geq 0 \\
1 \qquad + x_2 + x_3 &\geq 0 \\
- x_3 &\geq 0
\end{aligned}
$$

V-representation:

$$v_1 = (-1,1,0), \quad v_2 = (-1,-1,0), \quad v_3 = (1,-1,0),$$

$$v_4 = (1,1,0), \quad v_5 = (0,0,-1)$$

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$$

- Vertex enumeration: H-representation $\Rightarrow$ V-representation
- Convex hull problem: V-representation $\Rightarrow$ H-representation
- Solution methods: double description(cdd) and reverse search(lrs)

Who uses vertex enumeration?

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- Goal: parallelize *lrs* for multicore workstations using existing code

Case study: MIT problem

PHYSICAL REVIEW B

CONDENSED MATTER

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Ground states of a ternary fcc lattice model with nearest- and next-nearest-neighbor interactions

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(Received 9 September 1993)

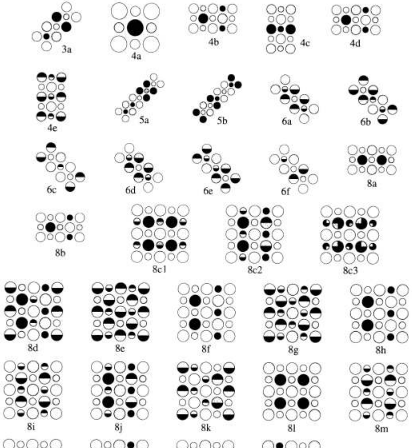
The possible ground states of a ternary fcc lattice model with nearest- and next-nearest-neighbor pair interactions are investigated by constructing an eight-dimensional configuration polytope and enumerating its vertices. Although a structure could not be constructed for most of the vertices, 31 ternary ground states are found, some of which correspond to structures that have been observed experimentally.

Case study: MIT problem

large problems. The drawback of the method is that many duplicates of the same vertex can be generated when degeneracy is present. While both methods successfully generated all vertices of the polytope, the double description method seems to be more appropriate for this computation because of the high degeneracy and moderate size of the inequality system. For larger systems, however, the reverse search method may become the only feasible algorithm for vertex enumeration.

III. RESULTS

The ground-state polytope we found is highly degenerate and consists of 4862 vertices in the eight-dimensional space spanned by the correlation functions. Some of the vertices found correspond to structures that can be transformed into each other by permutations of the A, B, and C species. If these are considered to be the same structure, the total number of distinct structures is



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- In 2012:

cddr+	*lrs*	*mplrs*					
		cores=8		cores=16		cores=32	
secs	secs	secs	su	secs	su	secs	su
368	496	99	5.0	44	11.2	26	19

Table: *mai64*: Opteron 6272, 2.1GHz, 64 cores, speedups(su) on *lrs*

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Table: *mai64*: Opteron 6272, 2.1GHz, 64 cores, speedups(su) on *lrs*

- 32-core speedup of *plrs* on 1993 *mplrs*: about 140,000 times! (processor=110 $\times$ 1300=software)

More cores

Name	lrs (mai20)	mplrs secs/efficiency 96 cores	128 cores	160 cores	192 cores	256 cores	312 cores
c40	10002 1	329 .48	247 .48	203 .46	179 .44	134 (.44)	129 (.37)
perm10	2381 1	115 .34	94 .31	85 .28	96 .20	64 (.23)	61 (.20)
mit71	21920 1	686 .54	516 .54	412 .54	350 .53	231 (.60)	205 (.55)
bv7	9040 1	302 .49	229 .49	184 .49	158 .47	98 (.57)	88 (.52)
cp6	1774681 1	56700 .63	43455 .62	34457 .63	28634 .63	18657 (.72)	15995 (.69)

Table: efficiency = speedup/number of cores (*mai* cluster)

Even more cores ...

Name	mplrs				
	1 core	300 cores	600 cores	900 cores	1200 cores
c40	17755	89	49	43	44
	1	.66	.60	.46	.34
mit71	36198	147	80	63	49
	1	.82	.75	.64	.62
bv7	10594	48	27	27	29
	1	.73	.65	.44	.30
cp6	2400648	9640	4887	3278	2570
	1	.83	.82	.81	.78

Table: *Tsubame2.5* at Tokyo Institute of Technology: secs/efficiency

Reverse Search (A. & Fukuda, '91)

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  - Generate all planar spanning tress on a given set of points.
  - Generate all the cells or vertices of an arrangement of lines planes, or hyperplanes.

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  - Generate all vertices of a convex polyhedron

Reverse Search (A. & Fukuda, '91)

- Space efficient technique to list unstructured discrete objects
- Typical Problems:
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- Reverse search is defined by an adjacency oracle and a local search function

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- f defines a spanning tree on G rooted at v^*
- Reverse search generates this tree starting at v^*

Example - Problem

Problem:
Generate permutations of $\{1, 2, .., n\}$

Input:
$n = 4$

Output:

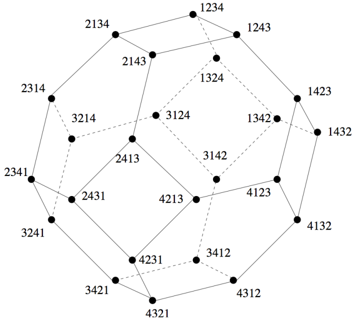
$$
\begin{array}{cccccc}
(1,2,3,4) & (1,2,4,3) & (1,3,2,4) & (1,3,4,2) & (1,4,2,3) & (1,4,3,3) \\
(2,1,3,4) & (2,1,4,3) & (2,3,1,4) & (2,3,4,1) & (2,4,1,3) & (2,4,3,1) \\
(3,1,2,4) & (3,1,4,2) & (3,2,1,4) & (3,2,4,1) & (3,4,1,2) & (3,4,2,1) \\
(4,1,2,3) & (4,1,3,2) & (4,2,1,3) & (4,2,3,1) & (4,3,1,2) & (4,3,2,1)
\end{array}
$$

Example - Adjacency Oracle

$\{\pi_1, \pi_2, ..., \pi_n\} is a permutation of \{1, 2, .., n\}$

$Adj(\pi, i) = (\pi_1, \pi_2, ..., \pi_{i-1}, \pi_{i+1}, \pi_i, ...\pi_n)$ for $i = 1, 2, ..., n-1$.

Note: $\Delta = n - 1$

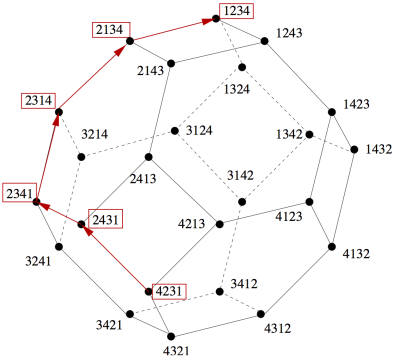


Example - Local Search

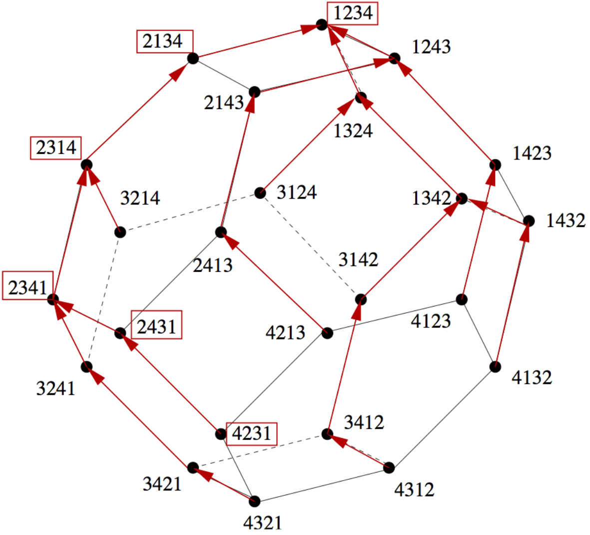
Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$
Target: $(1, 2, \ldots, n)$

$$f(\pi) = (\pi_1, \pi_2, \ldots, \pi_{i-1}, \pi_{i+1}, \pi_i, \ldots, \pi_n)$$

where i is the smallest index for which $\pi_i > \pi_{i+1}$.



Example - Reverse Search Tree

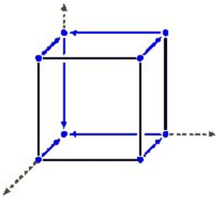


Reverse Search - Pseudocode

Algorithm 1 reverseSearch(v^*, Δ, Adj, f)

```
repeat
    v ← v*  j ← 0
    while j < Δ do
        j ← j + 1
        if f(Adj(v, j)) = v then
            v ← Adj(v, j)                forward step
            print v
            j ← 0
        end if
    end while
    if v ≠ v* then
        (v, j) ← f(v)                    backtrack step
    end if
until v = v* and j = Δ
```

Reverse search for vertex enumeration-I



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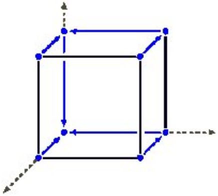

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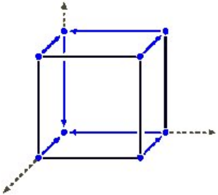
Reverse search for vertex enumeration-I



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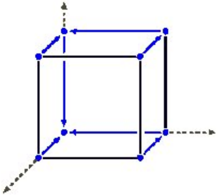

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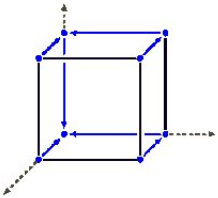

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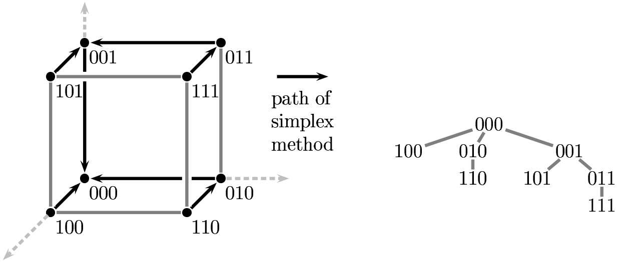
- Simplex method gives a path from any vertex to the optimum vertex
- lrs is a C implementation available on-line

Reverse search for vertex enumeration-II

http://cgm.cs.mcgill.ca/ avis/C/lrs.html



(a) The "simplex tree" induced by the objective $(-\sum x_i)$.

(b) The corresponding reverse search tree.

Reverse Search: features for parallelization

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- Each vertex is reported once and may be discarded afterwards
- Subtrees may be enumerated independently without communication
- Subtree size may be estimated by Hall-Knuth estimator

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Extension to allow :

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Extension to allow :

- all subtrees to be listed at some fixed depth
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- Additional parameters:
  - *maxd* is the depth at which forward steps are terminated.
  - *mind* is the depth at which backtrack steps are terminated.
  - d is the depth of subtree root v^*.

Extended Reverse Search - Pseudocode

Algorithm 2 extendedReverseSearch($v^*, \Delta, Adj, f, d, maxd, mind$)

repeat
  $v \leftarrow v^*$ $j \leftarrow 0$
  while $j < \Delta$ and $d < maxd$ **do**
    $j \leftarrow j + 1$
    if $f(Adj(v, j)) = v$ **then**
      $v \leftarrow Adj(v, j)$ *forward step*
      print v
      $j \leftarrow 0$
      $d \leftarrow d + 1$
    end if
  end while
  if $v \neq v^*$ **then**
    $(v, j) \leftarrow f(v)$ *backtrack step*
    $d \leftarrow d - 1$
  end if
until ($d = mind$ or $v = v^*$) and $j = \Delta$

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- Use available cores on user machine 'automatically'
- Reuse existing *lrs* code (8,000+ lines!)

Naive Parallel Reverse Search: 3 phases

- Phase 1: (single processor)
  - Generate the reverse search tree T down to a fixed depth *init_depth*.
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  - Schedule threads from L using subtree enumeration feature.
  - Use parameter *max_threads* to limit number of parallel threads.
  - Direct output to shared output stream.
- Phase 3: (partial parallelization)
  - Wait until all children threads terminate.

Parallel Reverse Search - Pseudocode

Algorithm 3 parallelReverseSearch($v^*, \Delta, Adj, f, id, mt$)

```
num_threads ← 0
redirect output to a list L                                   Phase 1
extendedReverseSearch(v*, Δ, Adj, f, 0, id, 0)
remove all v ∈ L with depth(v) < id and output v
while L ≠ ∅ do
   if num_threads < mt then
      remove any v ∈ L                                        Phase 2
      num_threads ← num_threads + 1
      extendedReverseSearch(v, Δ, Adj, f, depth(v), ∞, depth(v))
   end if
end while
while num_threads > 0 do
   wait for termination signal
   if L ≠ ∅ then
      wait until a termination signal is received
      extendedReverseSearch(v, Δ, Adj, f, depth(v), ∞, depth(v))
   else
      num_threads ← num_threads − 1                           Phase 3
   end if
end while
```

plrs (Implemented by Gary Roumanis)

A portable parallel implementation of *lrs* derived from the parallel reverse search algorithm.

Architecture:

- Light C++ wrapper around *lrs*.
  - Leverage *lrs's* restart feature.
  - Use portable g++ compiler.
- Multi-producer and single consumer.
  - Producer threads traverse subtrees of the reverse search tree, appending nodes to a lock-free queue.
  - Consumer thread removes nodes from shared queue and concatenates to unified location.
- Leverage open source Boost library for atomic features.
  - Ensures portability, maintainability and strong performance.

3 Phases: CPU utilization

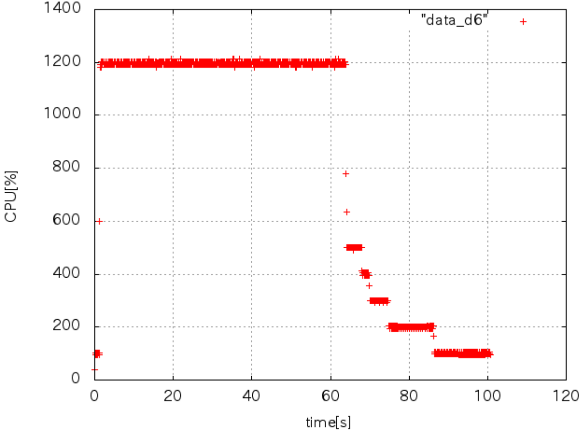
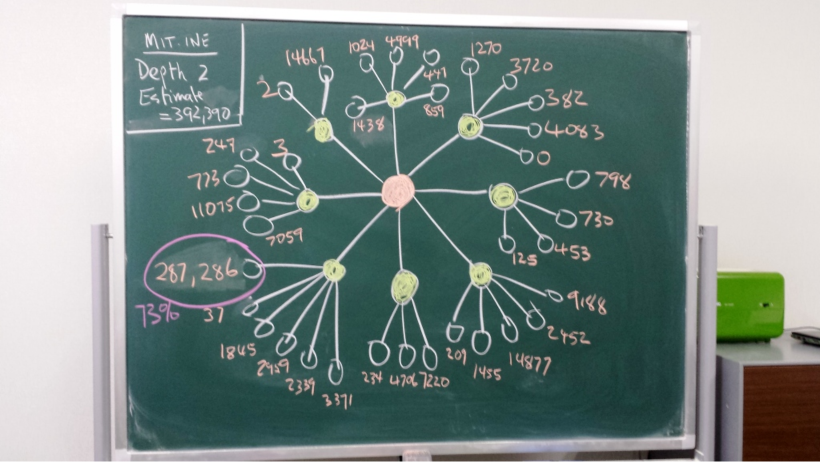


Figure: Input file: mit, *id* = 6, cores=12

Estimates at depth 2: mit



Initial depth variation: mit

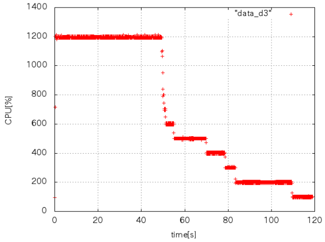


Figure: $id = 3$, $L = 127$, 124 secs

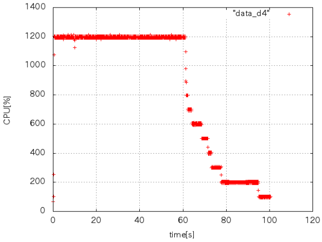


Figure: $id = 4$, $L = 284$, 105 secs

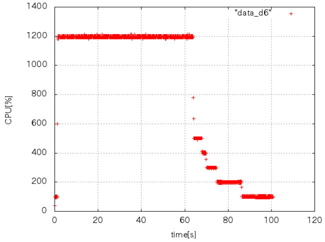


Figure: $id = 6$, $L = 1213$, 105 secs

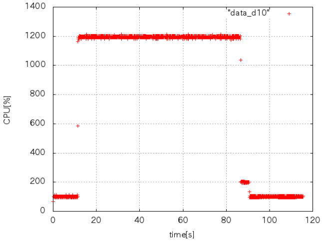


Figure: $id = 10$, $L = 7985$, 125 secs

plrs: limitations

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  - Please come back for part 2!